Resolution of Vectors

Apparatus: Force table, pulleys, mass hangars, various slotted masses, string, scientific calculator (set to degree mode), protractor.

Objectives: To show that any vector may be resolved into components which have the same effect as the original vector. To determine the resultant for several coplanar vectors experimentally, graphically, and analytically.

Discussion: A *vector* is a physical quantity that has two properties: *magnitude* and *direction*. Some vectors that you are probably familiar with are force (***F***), velocity (***v***), displacement (***x***), and acceleration (***a***).

Suppose we journey across campus, following the sidewalks, walking 100 meters north, 275 meters east, and 200 meters north. How far have we gone? Even though we have walked 575 meters, we are only slightly over 400 meters from where we started. In order to state the situation precisely, a physicist would frame this situation in terms of vectors.

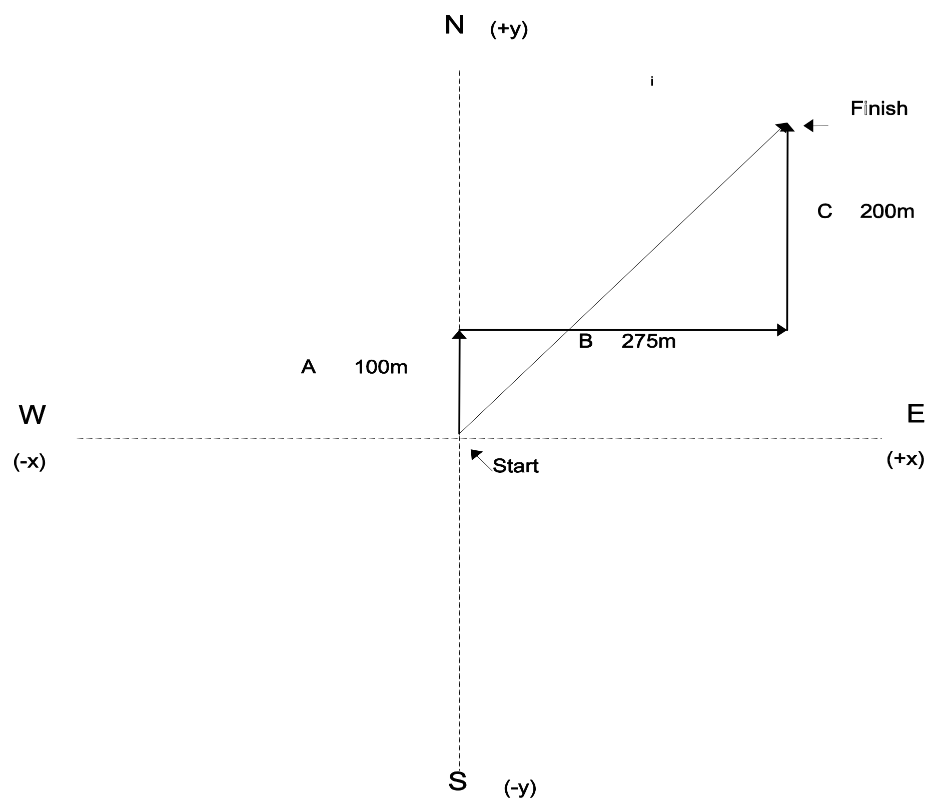

Figure 1. A walk across campus.

Since we are moving in a *plane* only two coordinates are necessary to describe our motion: N-S and E-W (we could just as easily use x and y). It would be useful to define some *origin* for our coordinate system - which we will choose to be the point at which we began our hypothetical journey. As a matter of convenience I will assign this point the coordinates $x = 0$, $y = 0$. If we walk 100 meters north, for instance, we have walked 100 meters in the $+y$ direction.

What we would like to know is our *displacement* from the origin at the end of our walk. Displacement is the shortest distance between two points, i.e., the path we would have followed had we decided to make a beeline and cut straight across campus to our destination. This total displacement may be represented by a vector which will point from where we started to where we ended up. To compute the displacement between these two points we should consider each segment of our journey as a displacement vector, break each of these vectors into components, and add up all of the components. The sum of the components will be another vector representing total displacement.

$$\vec{A} = 0_x + 100_y$$
$$\vec{B} = 275_x + 0_y$$
$$\vec{C} = 0_x + 200_y$$
$$\therefore$$
$$\vec{R} = 275_x + 300_y$$

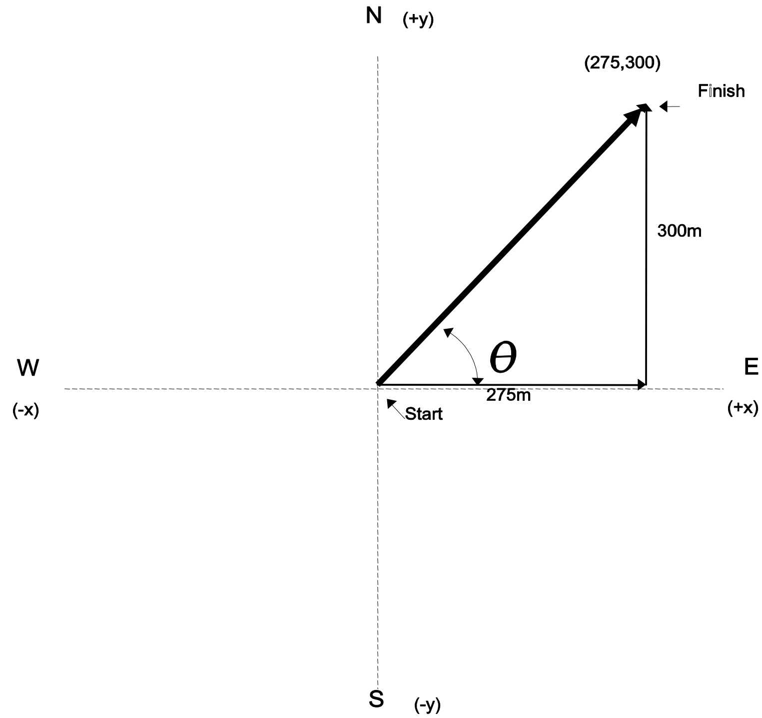

Figure 2.

During the first leg of our journey we traveled 100 meters due north. In our coordinate system this corresponds to a vector that's components are 0 meters in the x direction and 100 meters in the y direction, or $\boldsymbol{A} = 0_x + 100_y$. We then traveled 275 meters east, corresponding to the vector $\boldsymbol{B} = 275_x + 0_y$. Finally, we traveled 200 meters due north, i.e., $\boldsymbol{C} = 0_x + 200_y$. Adding the x components of ***A***, ***B*** and ***C*** will give the x component of a *resultant* vector, and adding the y components of ***A***, ***B*** and ***C*** together will give the y component of the resultant vector. This vector, ***R***, contains the components of our displacement vector.

Why should we be interested in this bit of manipulation? The components of ***R*** (275, 300) represent a point in the Cartesian plane so our displacement vector ***R*** points from the origin to this point.

Notice that the components of ***R*** form the sides of a right triangle while ***R*** itself forms the hypotenuse of the right triangle. Recall that for any right triangle:

$$a^2 + b^2 = c^2 \quad \textit{Pythagorean Theorem}$$

where *c* is the length of the hypotenuse. Hence, the length of ***R*** is:

$$R = \sqrt{R_x^2 + R_y^2}$$
$$R = \sqrt{275\ m^2 + 300\ m^2}$$
$$R \approx 407m$$

The direction of ***R***, by convention, is specified as an angular displacement (θ) from the positive *x* axis. Again, referring to the right triangle in Figure 2:

$$tan(\theta) = \frac{opp}{adj} = \frac{y}{x} \_ \theta = tan^{-1}\frac{y}{x}$$
$$sin(\theta) = \frac{opp}{hyp} = \frac{y}{R} \_ \theta = sin^{-1}\frac{y}{R}$$
$$cos(\theta) = \frac{adj}{hyp} = \frac{x}{R} \_ \theta = cos^{-1}\frac{x}{R}$$

The result is the same regardless of which way we do the calculation since θ is the same angle in each case. <u>In this example, $\theta \approx 47.5^0$</u>. Our displacement from the origin, the vector ***R***, may be specified in one of two ways:

$$\vec{R} = 275\ m_x + 300\ m_y \quad \textit{(component form)}$$
$$\vec{R} = 407m\ @\ 47.5^0 \quad \textit{(polar vector form)}$$

Both forms contain magnitude and direction and may be used interchangeably. We could have walked 275 meters east and 300 meters north, or walked 407 meters 47.5^0 north of east and reached the same point as we did via our original path.

In this example we always walked in a direction that coincided with a coordinate axis, i.e., due north and east. This meant that each individual displacement had only one component (e.g., north was along the +*y* axis). What happens if we decide to walk in some direction other than along a coordinate axis, say northeast?

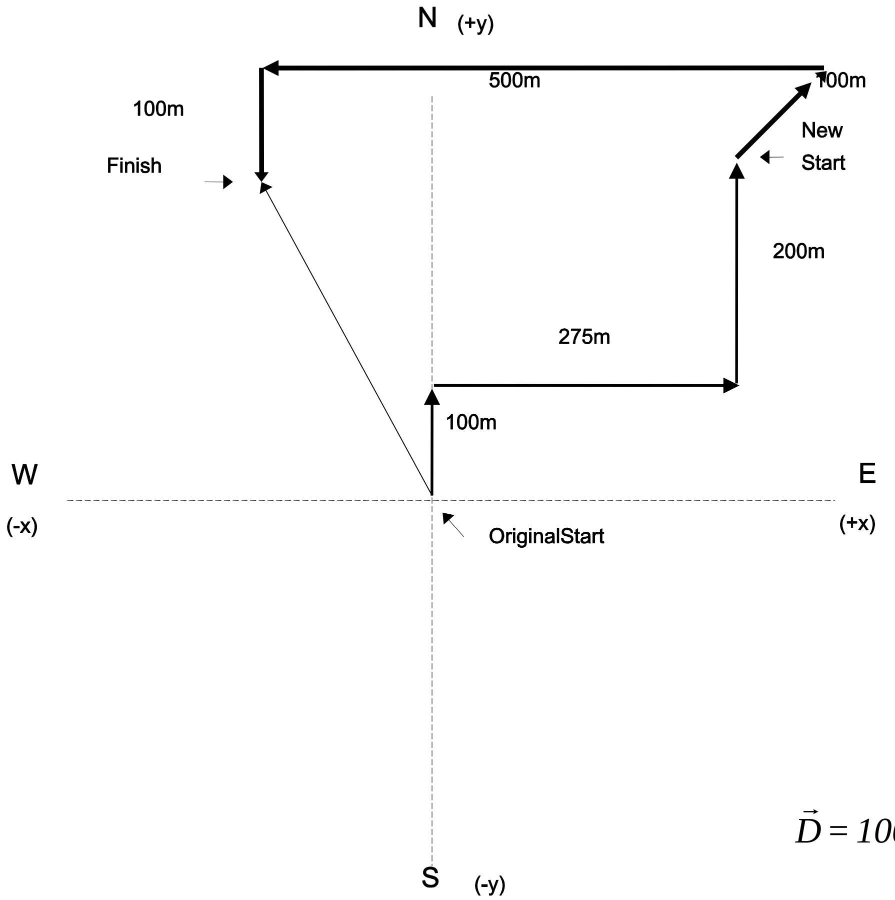

Figure 3.

$$\vec{A} = 0_x + 100_y$$
$$\vec{B} = 275_x + 0_y$$
$$\vec{C} = 0_x + 200_y$$
$$\vec{D} = 100\,(\cos 45^0\,)_x + 100\,(\sin 45^0\,)_y$$
$$\vec{E} = -500_x + 0_y$$
$$\vec{F} = 0_x - 100_y$$
$$\therefore$$
$$\vec{R} = -154_x + 271_y$$

Suppose we start walking from where we ended up in the last example (the point 275, 300). We proceed 100 meters northeast, 500 meters due west and 100 meters south as in Figure 3. What is our displacement from our original starting point?

Again, by adding the components of vectors ***A*** - ***F*** we arrive at a displacement of 154 meters in the -*x* direction and 271 meters in the *y* direction. The only difficulty here lies computing the components of ***D***, the vector representing the only leg of our journey not to lie along a coordinate axis.

Just as we can combine components of a vector to get magnitude and direction, we can separate magnitude and direction into a set of components. Our direction of travel along ***D*** is northeast, halfway between north and east. This corresponds to an angle θ of 45^0 from the $+x$ (east) axis.

Figure 4 shows the method used to resolve ***D*** into components $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$, the *projections* of ***D*** along the coordinate axes. The magnitudes of the components may be computed by combining vector properties with a bit of trigonometry.

Vectors may be displaced in any direction as long as their magnitude and direction are preserved.

Consider Figure 5. Each of the vectors $\boldsymbol{A}_1$ - $\boldsymbol{A}_5$ is equivalent. $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are displacements of $\boldsymbol{A}_1$ along the *x* and *y* axes. $\boldsymbol{A}_4$ is a displacement of $\boldsymbol{A}_1$ through the origin of the coordinate system. $\boldsymbol{A}_5$ is a displacement of $\boldsymbol{A}_1$ from the first to the fourth quadrant.
Since the magnitude and direction of $\boldsymbol{A}_1$ has been preserved through each shift, each of these vectors are the same.

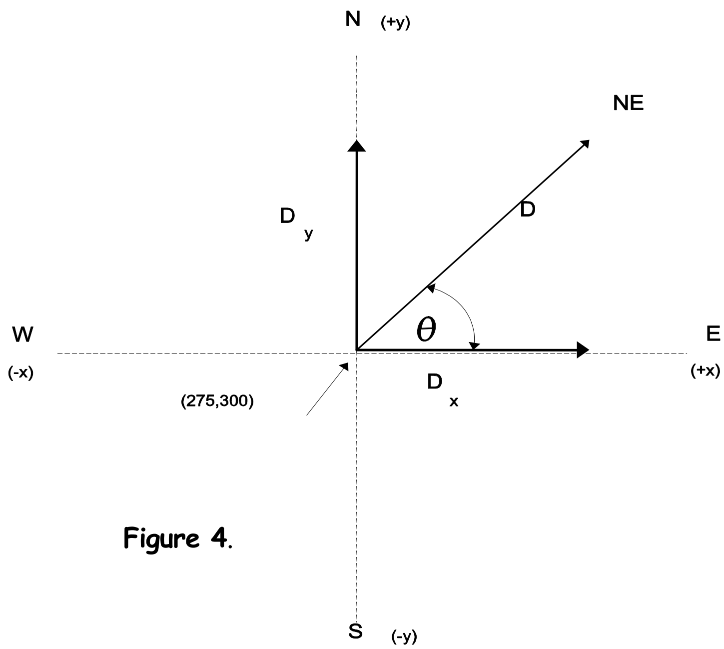

Figure 4.

Let's go back to Figure 4. Since we can displace $\boldsymbol{D}_y$ or $\boldsymbol{D}_x$ any way we choose as long as we preserve their magnitude and direction, we can slide $\boldsymbol{D}_y$ along the x axis until its tip is touching the tip of ***D*** (Figure 6). By doing so we have created a right triangle with $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$ as the sides adjacent and opposite to θ, and ***D*** as the hypotenuse.

Using the identities:

$$\sin\theta = \frac{opp}{hyp} = \frac{\vec{D}_y}{R} \_ \vec{D}\sin(\theta) = \vec{D}_y$$

$$\cos\theta = \frac{adj}{hyp} = \frac{\vec{D}_x}{R} \_ \vec{D}\cos(\theta) = \vec{D}_x$$

we find that the component of ***D*** in the *y* direction is equal to ***D***sin(θ) and the component in the *x* direction is equal to ***D***cos(θ). The angle θ in our example is 45^0. The sine and cosine of 45^0 are .707, so vector ***D*** may be expressed in terms of components as $\boldsymbol{D} \approx 71_x$ m + 71_y m.

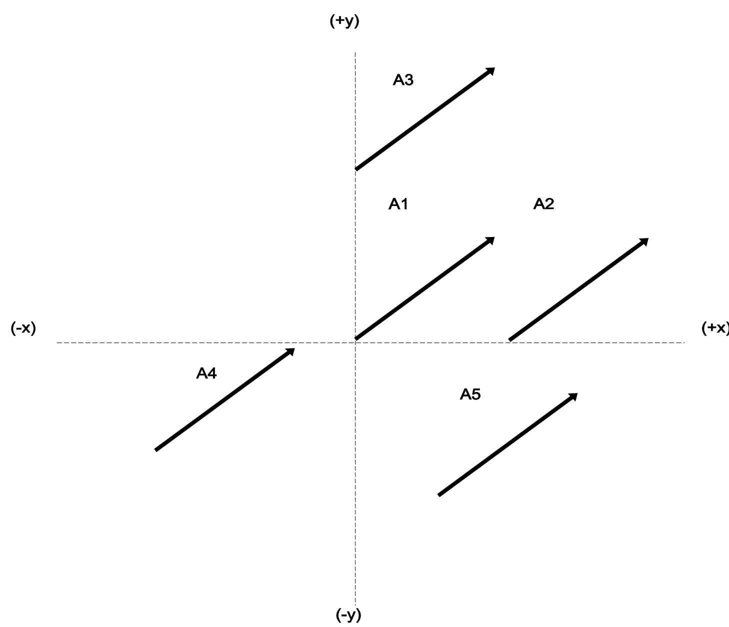

Figure 5. Vectors $\boldsymbol{A}_1$ - $\boldsymbol{A}_5$ are equivalent.

Our final displacement has components of 154 meters in the -*x* direction and 171 meters in the *y* direction. This may be represented as in Figure 7. Notice that $\boldsymbol{R}_y$ has been shifted left along the *x* axis so that $\boldsymbol{R}_x$, $\boldsymbol{R}_y$, and $\boldsymbol{R}$ form the sides of a right triangle. The magnitude and direction of $\boldsymbol{R}$ may be found by:

$$R = \sqrt{R_x^2 + R_y^2}$$

$$R = \sqrt{-154\ m^2 + 271\ m^2}$$

$$R \approx 312m$$

$$tan(\theta) = \frac{opp}{adj} = \frac{R_y}{R_x} \_ \theta = tan^{-1}\frac{271\ m}{-154\ m}$$

$$\approx -60^0$$

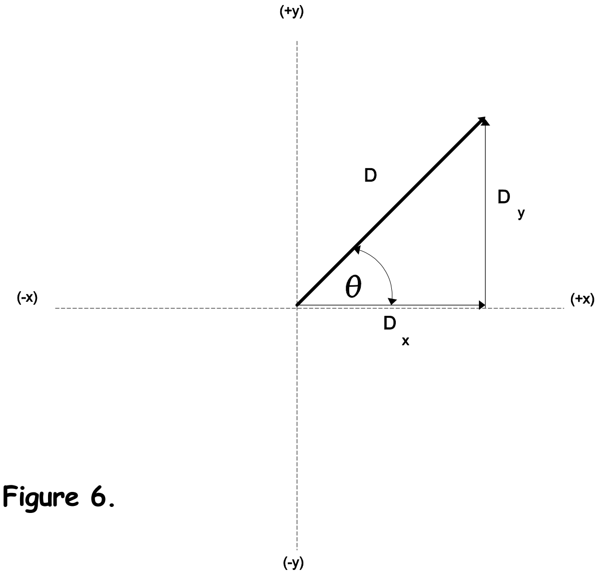

Figure 6.

R has a magnitude of 312 meters and is displaced -60^0 (clockwise) from the -*x* axis or 120^0 from the +*x* axis.

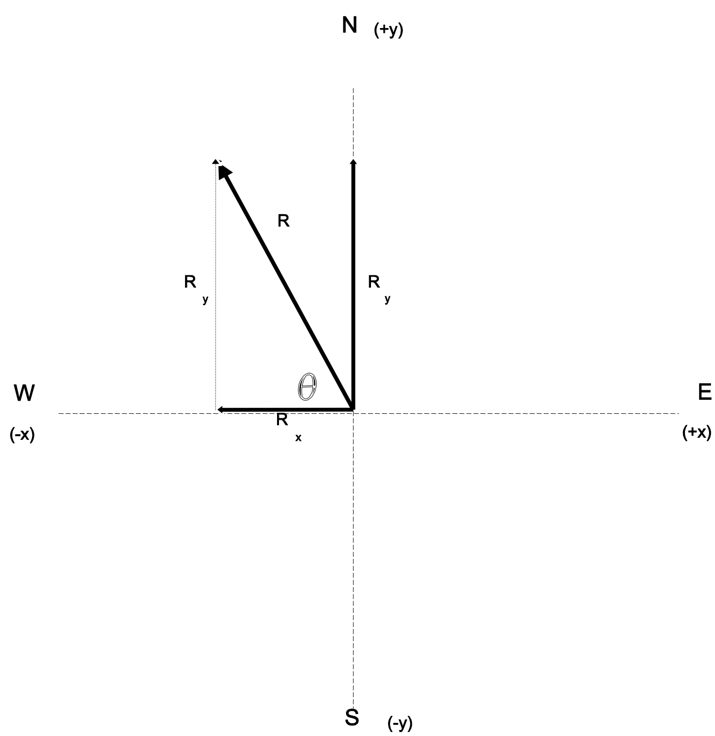

Figure 7.

In the preceding examples we tackled vector addition by breaking each vector into components, adding the components, then recombining the components into a resultant vector. Another method of vector addition is the graphical method. This method makes use of the fact that vectors may be displaced anywhere in the Cartesian plane as long as their magnitude and direction are preserved. Figures 8 and 9 are examples of graphical vector addition. In figure 8, vector ***A*** lies in the 1st quadrant and vector ***B*** lies in the 4th. To add ***A*** and ***B*** graphically one shifts the tail of ***B*** along ***A*** until the tip of ***A*** touches the tail of ***B***. The resultant of these two vectors, ***A*** + ***B***, may then be found by extending a line from the origin to the tip of ***B***. Notice the same result would be achieved by sliding the tail of ***A*** to the tip of ***B*** (try it).

What does this tell you about the order of addition in vectors?

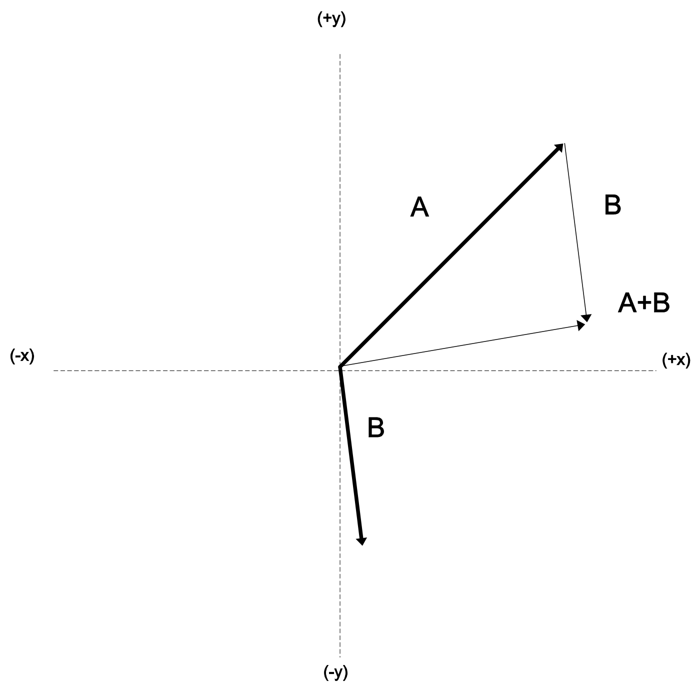

Figure 8.

Figure 9 is a more complicated example involving four vectors. The order of addition used was ***A*** + ***B*** + ***C*** + ***D***. Reproduce figure 9 in your lab notebook and see what happens if you change the order of this operation. What result do you expect?

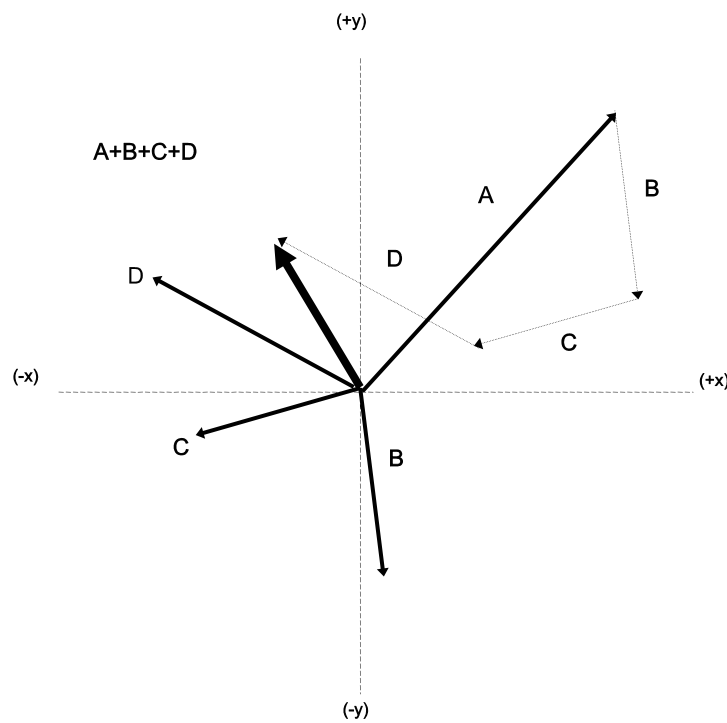

Figure 9.

Procedure: In this exercise you will gain experience resolving vectors. You will be using an apparatus known as a force table. The force table is a device that allows one to measure angles. The vectors you will be dealing with are the *mg* (weight) forces of masses suspended from strings as shown in figure 10. The magnitude of the forces may be found from adding up the masses. The directions may be read directly from the graduated scale around the edge of the force table.

Your lab instructor will give you two sets of vectors to work with. Your first objective will be to show that a given force may be broken up into components that exactly duplicate the force. Your second objective will be to resolve a set of three forces into a single equivalent force. For each of these objectives you are to a) compute an algebraic solution, b) solve the problem graphically, c) demonstrate that your solutions are valid on the force table.

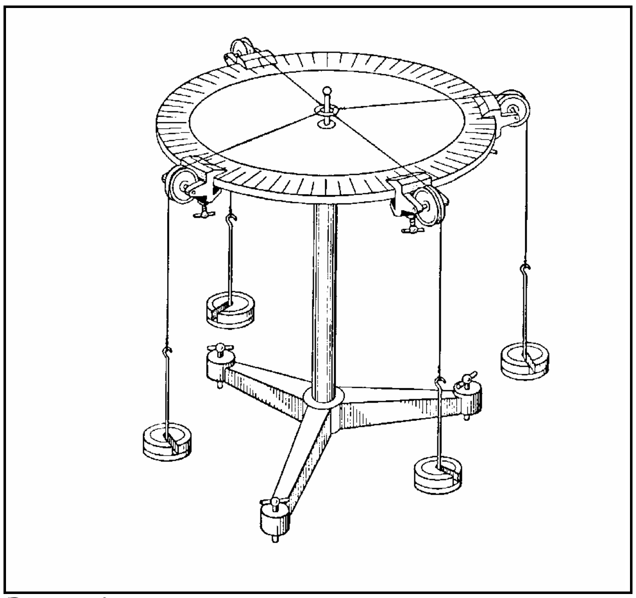

Figure 4.