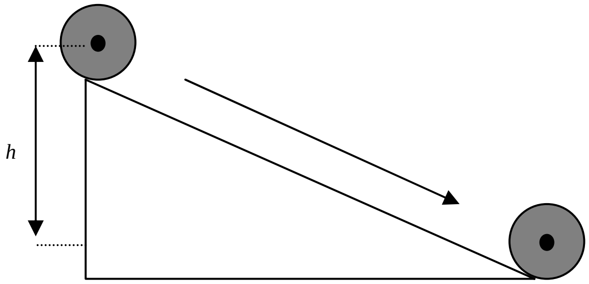
Translation/Rotation & Energy

Consider a solid cylinder of radius *R* that rolls without slipping down an incline from some initial height *h*. The linear velocity of the cylinder at the bottom of the incline is v_{cm} and the angular velocity is ω.



- If the cylinder starts from rest, all of its subsequent kinetic energy comes from gravitational potential energy.
- Because the cylinder is both translating and rotating as it moves down the plane, some of this initial energy goes into rotation and some goes into translation.
- This means that the linear velocity of the cylinder at the bottom of the plane is slower than it would be if the cylinder slid down the plane without rotating. Energy is still conserved, but the initial potential energy is now converted into two types of kinetic energy.

$$PE_i = KE_{f_{rot}} + KE_{f_{trans}}$$

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}{}^2$$

For pure rolling motion (i.e., no slipping) $v_{cm} = R\omega$.

$$mgh = \frac{1}{2}I\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

For a solid cylinder rotating about a symmetry axis down the length of the cylinder,

$I = \frac{1}{2}MR^2$. Inserting this into the equation above yields:

$$mgh = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

$$gh = \frac{1}{2}\left(\frac{1}{2}R^2\right)\frac{v_{cm}{}^2}{R^2} + \frac{1}{2}v_{cm}{}^2 \rightarrow gh = \frac{1}{4}v_{cm}{}^2 + \frac{1}{2}v_{cm}{}^2 \therefore v_{cm} = \sqrt{\frac{4}{3}gh}$$

Note that this is less than $\sqrt{2gh}$ that would be the case for purely translational motion. Notice also that the result does not depend upon either the mass or radius of the cylinder.

We can also solve for angular velocity.

$$mgh = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + \frac{1}{2}m(R\omega)^2$$

$$gh = \frac{1}{2}\left(\frac{1}{2}R^2\right)\omega^2 + \frac{1}{2}(R\omega)^2 \rightarrow gh = \frac{1}{4}R^2\omega^2 + \frac{1}{2}R^2w^2 \therefore gh = \frac{3}{4}R^2\omega^2 \therefore \omega = \sqrt{\frac{4}{3}\frac{gh}{R^2}}$$

What is the ratio of rotational to translational energy?

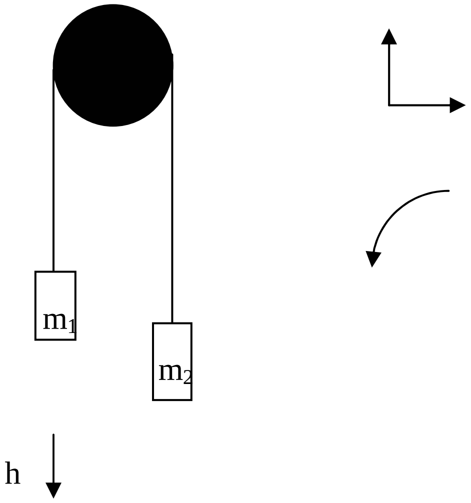
$$\frac{KE_r}{KE_t} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}{}^2} = \frac{\frac{1}{4}}{\frac{1}{2}} = 50\% \text{ (rotation has half the energy of translation)}$$

What percentage of the total kinetic energy goes into rotation?

$$\frac{KE_r}{KE_{total}} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}{}^2 + \left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2} = \frac{\frac{1}{4}}{\frac{3}{4}} = 33\%$$

You should perform the same analysis for a both a hoop and a sphere of the same mass *M* and the same radius *R*. Based on your calculations, which reaches the bottom first in a three-way race, a hoop, a solid sphere, or a solid cylinder?

Example 1. Find the velocity of the following system ($m_1 > m_2$) after it has moved through some height h. The pulley has mass M, radius R, and $I=MR^2$. Note $T_1 \neq T_2$.



- $E_i = E_f$
- $\frac{1}{2}m_1 v_i^2 + \frac{1}{2}m_2 v_i^2 + \frac{1}{2}I\omega_i^2 + m_1 g h_i + m_2 g h_i = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}I\omega_f^2 + m_1 g h_f + m_2 g h_f$
- $m_1 g h_i = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}I\omega_f^2 + m_2 g h_f$

Recall: $\omega = \frac{v}{R}$, $I = MR^2$

$$\therefore m_1 g h = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}(MR^2)\frac{v_f^2}{R^2} + m_2 g h$$

$$m_1 g h = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + \frac{1}{2}M v_f^2 + m_2 g h$$

$$2(m_1 g h - m_2 g h) = (m_1 + m_2 + M)v_f^2$$

$$v = \sqrt{2\frac{m_1 - m_2}{m_1 + m_2 + M}gh}$$

Example 2. A time-honored tradition in Appalachia is *boulder trundling* where a large round rock is chucked off a tall cliff just for the sake of mischief. Let's say that high school kids heave a large (200kg, 0.6 meters in diameter), roughly spherical boulder over the edge of a 50 meter tall strip mine headwall. It falls down the cliff face, hits the ground and begins rolling 50 meters down a steep incline before crashing into a mining company jeep. What is the velocity of the boulder just as it hits the jeep?

It is tempting to just divide up the initial potential energy into translational and rotational kinetic energy. But not so fast! Here we have an interval in which the sphere is translating without rotating. Does this make a difference in the final result?

During the first part of the problem, before the boulder hits the ground, all of the initial potential energy is converted into translational kinetic energy.

$$v_{boulder} = \sqrt{2gh} = 31.3 m \cdot s^{-1}$$

After the boulder starts rolling down the hill whatever energy it has is converted to both translational and rotational kinetic energy. The total energy at the moment the boulder begins rolling down the hillside is composed of the kinetic energy from the fall down the cliff (or the potential energy from the top of the cliff with respect to the ground), and the potential energy with respect to the jeep.

$$KE + PE = KE_{trans} + KE_{rot}$$

$$\frac{1}{2}mv_i^{\ 2} + mgh = \frac{1}{2}mv_f^{\ 2} + \frac{1}{2}I\omega^2$$

$$\frac{1}{2}v_i^{\ 2} + gh = \frac{1}{2}v_f^{\ 2} + \frac{1}{2}\left(\frac{2}{5}R^2\right)\frac{v_f^{\ 2}}{R^2}$$

$$\frac{1}{2}v_i^{\ 2} + gh = \frac{1}{2}v_f^{\ 2} + \frac{1}{5}v_f^{\ 2}$$

$$\frac{1}{2}v_i^{\ 2} + gh = \frac{7}{10}v_f^{\ 2} \rightarrow 980 m \cdot s^{-1} = \frac{7}{10}v_f^{\ 2} \therefore v_f = 37.4 m \cdot s^{-1}$$

Now let's consider what would have happened if we'd just rolled the boulder down a 100 meter tall incline:

$$mgh = \frac{1}{2}mv_f^{\ 2} + \frac{1}{2}I\omega^2 \rightarrow gh = \frac{7}{10}v_f^{\ 2} \therefore v_f = 37.4m \cdot s^{-1}$$

So there is no difference in the final velocity of the boulder. What gives?

- Since energy is conserved in this example the total energy available to the boulder is the potential energy of 100 meters in height. It doesn't really matter when the boulder begins to roll in terms of its final linear velocity. The only difference will be in the rate of angular acceleration.
- In which scenario would it be greater?
- Would this problem have a different final answer if the boulder was rolled off the cliff and rotated as it dropped to the ground?

Finally, older Jeeps are made out of some pretty stout metal. If the boulder whacks the 1900 kg Jeep without denting it significantly, and both move 1 meter before coming to rest, what is the average force exerted on the occupants of the Jeep during the collision?

This is an inelastic collision:

$$m_{boulder}v_i + M_{jeep}v_i = (M + m)v_f$$

$$mv_i = (m + M)v_f \rightarrow v_f = \frac{m}{m + M}v_i \therefore v_f = 3.6m \cdot s^{-1}$$

This is the velocity of the Jeep and the boulder as they begin to move together. Since they come to rest their final velocity is zero.

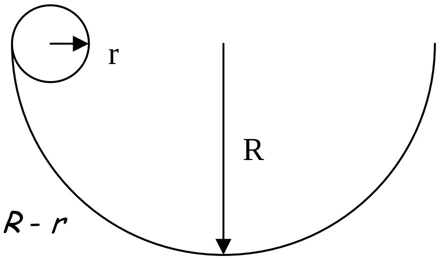
$$\bar{v} = \frac{(3.6 + 0)m \cdot s^{-1}}{2} = 1.8m \cdot s^{-1}$$

$$\bar{v}t = d \therefore \frac{d}{\bar{v}} = t = \frac{1m}{1.8m \cdot s^{-1}} = 0.56s$$

$$\overline{F}\Delta t = \Delta\vec{p} \therefore \overline{F} = \frac{\Delta p}{\Delta t} = 13500N$$

That'll leave a mark!

Example 3. *A* uniform solid sphere of radius *r* is placed on the inside surface of a hemispherical bowl of radius *R*. The sphere is released from rest at the edge of the bowl as shown below and rolls without slipping. Determine the translational speed of the sphere when it reaches the bottom of the bowl.



center of mass of the sphere is located at *R* - *r*

conserve energy: $E_i = E_f$

$$mg(R-r)_i + \frac{1}{2}m{v_i}^2 = mg(R-r)_f + \frac{1}{2}m{v_f}^2 + \frac{1}{2}I\omega^2$$

$$mg(R-r) = +\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2$$

$$v = r\omega \therefore mg(R-r) = +\frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2$$

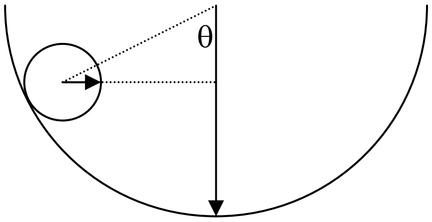
$$g(R-r) = \frac{7}{10}v^2 \ \therefore v = \sqrt{\frac{10}{7}g(R-r)}$$

What is the angular velocity of the sphere at the bottom of the bowl?

$$v = r\omega \therefore mg(R-r) = +\frac{1}{2}mr^2\omega^2 + \frac{1}{5}mr^2\omega^2 = \frac{7}{10}mr^2\omega^2$$

$$g(R-r) = \frac{7}{10}r^2\omega^2 \ \therefore \omega = \sqrt{\frac{10}{7}\frac{g(R-r)}{r^2}}$$

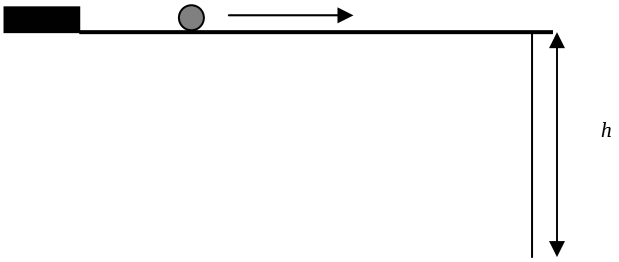
What is the angular velocity of the sphere if it is released at a random point along the edge of the bowl some angle θ to the vertical?



$$h = (R - r) - \left[(R - r)(\cos\theta)\right]$$

$$\therefore \omega = \sqrt{\frac{10}{7}\frac{g(R - r) - \left[(R - r)(\cos\theta)\right]}{r^2}}$$

Example 4. The projectile launchers we use in lab use a spring to supply the force needed to shoot a projectile across a room. Suppose that the projectile used is a solid metal sphere of mass 100 grams and radius 2 cm, the spring constant, *k*, is 500 N/m, the launcher is "cocked" by displacing the spring 10 cm from it's equilibrium position. The launcher is located on a level table. If the table is 1 meter in height, were does the projectile land?



We must first compute the energy available to this system. Initially this is potential energy stored in the spring of the launcher.

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}\left(500N \cdot m^{-1}\right)\left(0.1m\right)^2 = 2.5J$$

In the absence of any rotation this would result in a translational velocity of:

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_{cm}{}^2 \Rightarrow v_{cm} = \sqrt{\frac{kx^2}{m}} = 7.1m \cdot s^{-1}$$

In this case the initial potential energy of the spring is converted to translational kinetic energy and rotational kinetic energy. Since everything is conservative:

$$PE_i = KE_t + KE_r$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

We are interested in the velocity of the center of mass so:

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v^2}{R^2}\right)$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$$

$$\frac{1}{2}kx^2 = \frac{7}{10}mv_{cm}{}^2 \rightarrow \sqrt{\frac{5}{7}\frac{kx^2}{m}} = v_{cm} \approx 6.0 m \cdot s^{-1}$$

Note that the ratio of energies:

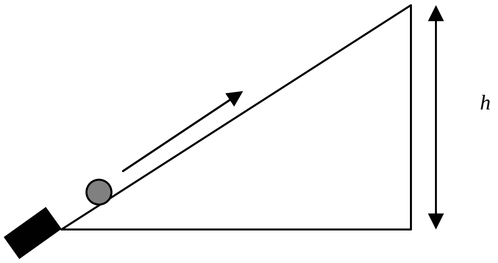
$$\frac{KE_r}{KE_{total}} = \frac{\frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}{}^2 + \left(\frac{2}{5}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2} = \frac{\frac{1}{5}}{\frac{7}{10}} \approx 29\%$$

would have allowed us to predict this since 71% of 2.5 Joules results in a translational velocity of about 6 m/s.

From here on this is a projectile motion problem with $v_0 \approx 6.0$ m/s and $y - y_0 = -1$ meter. The range is 2.7 meters.

The width of the table was not given and evidently does not matter. Why?

Example 5. The projectile launchers we use in lab use a spring to supply the force needed to shoot a projectile across a room. Suppose that the projectile used is a solid metal sphere of mass 100 grams and radius 2 cm, the spring constant, *k*, is 500 N/m, the launcher is "cocked" by displacing the spring 10 cm from it's equilibrium position. The launcher is located at the base of a ramp inclined at 30^0 to the horizontal up which the projectile rolls without slipping once it leaves the barrel of the launcher. If the ramp is 1 meter in height, were does the projectile land?



As before, we must first compute the energy available to this system. Initially this is potential energy stored in the spring of the launcher.

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}\left(500N \cdot m^{-1}\right)\left(0.1m\right)^2 = 2.5J$$

Since energy is conserved, if only translation were involved the ball would rise to height:

$$2.5J = mgh \therefore h = \frac{2.5J}{\left(0.1kg\right)\left(9.8m \cdot s^{-2}\right)} = 2.6m$$

So we are certain that the projectile would end up somewhere right of the ramp if *only* translation were involved. Since less than 30% of the total energy goes into rotation, we are pretty safe in assuming that the projectile will still clear the ramp. Can you verify this?

In this case the initial potential energy of the spring is converted to gravitational potential energy, translational kinetic energy and rotational kinetic energy. Since everything is conservative:

$$PE_i = PE_f + KE_t + KE_r$$

$$\frac{1}{2}kx^2 = mgh + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

We are interested in the velocity of the center of mass so:

$$\frac{1}{2}kx^2 = mgh + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v^2}{R^2}\right)$$

$$\frac{1}{2}kx^2 = mgh + \frac{1}{2}mv^2 + \frac{1}{5}mv^2$$

$$\frac{1}{2}kx^2 - mgh = \frac{7}{10}mv^2$$

At a height of 1 meter:

$$\frac{1}{2}kx^2 - mg = \frac{7}{10}mv^2 \Rightarrow \sqrt{\frac{5}{7}\frac{kx^2}{m} - \frac{10}{7}g} = v_{cm} \approx 4.7m \cdot s^{-1}$$

From here on this is a projectile motion problem with $v_0 \approx 4.7$ m/s @ 30^0, and $y - y_0 = -$ 1 meter. The range of the projectile is about 3 meters.

- How would does this compare with the result derived in the pervious example?
- The range is still greater even with the energy that goes into rotation. There are two competing effects here: the increased launch angle and the energy that goes into rotation. Do you think that this might have an effect on the launch angle that yields maximum range?

Example 6. A thin hollow cylinder is released from rest side by side with a solid cylinder of identical mass and radius. Both roll without slipping down an incline 10 meters in height, across a flat area and up an identical incline on the other side of the flat. To what height do the respective cylinders rise?

a) What is the translational velocity of the hollow cylinder at the bottom of the incline?

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}{}^2$$

$$mgh = \frac{1}{2}I\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

$$mgh = \frac{1}{2}\left(mR^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

$$gh = \frac{1}{2}v_{cm}{}^2 + \frac{1}{2}v_{cm}{}^2 \therefore v_{cm} = \sqrt{gh} = 9.9m \cdot s^{-1}$$

b) What is the translational velocity of the solid cylinder at the bottom of the incline?

By the same reasoning $v_{cm} = \sqrt{\frac{4}{3}gh} = 11.4m \cdot s^{-1}$

As the two cylinders roll across the flat area energy is conserved in the absence of non-conservative forces so their velocities do not change. As each reaches the other side:

$$KE_{i_{rot}} + KE_{i_{trans}} = PE_f$$

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}{}^2 = mgh$$

For the hollow cylinder:

$$gh = \frac{1}{2}v_{cm}{}^2 + \frac{1}{2}v_{cm}{}^2 \therefore h = \frac{v_{cm}{}^2}{g} = 10m$$

For the solid cylinder:

$$gh = \frac{1}{4}{v_{cm}}^2 + \frac{1}{2}{v_{cm}}^2 \therefore h = \frac{3}{4}\frac{{v_{cm}}^2}{g} = 10m$$

- Does one cylinder reach this height before the other?
- What if the problem were modified so that the track didn't have a flat spot at the bottom. What bearing would this have on the outcome?