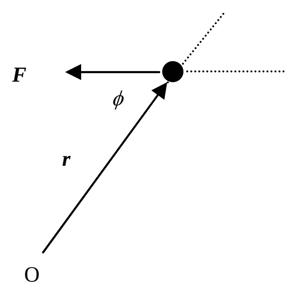
Torque – the Rotational Analog of Force

Torque is a force that causes a rotational acceleration of a rigid body about an axis or motion of a single particle relative to some fixed point.

- Torque is a vector.
- $\Gamma = \vec{r} \times \vec{F}$ or $rF \sin\theta$
- Torque comes from Latin and means "to twist".

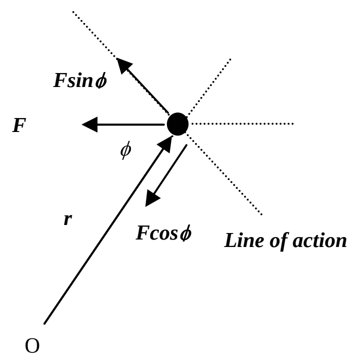
Consider a force ***F*** applied to a single particle in a plane given by position vector ***r*** relative to some fixed origin:

$$\Gamma = \vec{r} \times \vec{F} \qquad \text{(Definition of torque)}$$



- The angle between ***r*** and ***F*** is ϕ.
- The magnitude of the torque is ***rF***sinϕ.
- The product $\vec{r} \times \vec{F}$ is in a plane perpendicular to the plane containing ***r*** and ***F*** and in this example is out of the plane of the page.
- n this case the direction of the product of ***r*** and ***F*** is given by the *right hand rule*. If one points the fingers on their right hand in the direction or ***r*** and curls then in the direction of ***F*** the thumb points out of the plane of the page.
- Notice that either definition of torque "picks off" the component of ***F*** that is *perpendicular* to ***r***. If ***F*** and ***r*** are either parallel or antiparallel, no torque exists.

- The equation for magnitude of torque is often written $\Gamma = rF_{\perp}$ where $\boldsymbol{F}_{\perp}$ is $\boldsymbol{F}\sin\phi$. Note that this is equivalent to $\Gamma = r_{\perp}F$ where $\boldsymbol{r}_{\perp}$ is $\boldsymbol{r}\sin\phi$.
- The component of ***F*** defined by $\boldsymbol{F}\sin\phi$ lies along an imaginary line known as the *line of action*. It is this component of the force ***F*** in which we are interested. This is the component that supplies the torque.

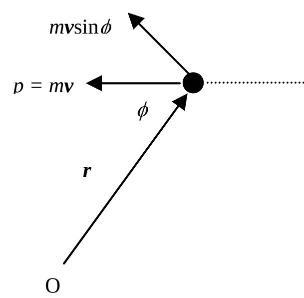


- Notice that in order to maximize the torque on a particle one ***F*** should be at right angles to ***r*** and the distance from the origin should be as great as possible.
- An extended rigid body may be thought of as being a system of such particles. The total torque on the system would be the sum of the torques on the individual particles.
- The SI unit of torque is a N · m. Notice that this is the same SI unit as *work*. What is the difference?
- The direction of torque is either + or -.
- *Mechanical Advantage* – trading force or torque for distance.

Intro to Angular Momentum

We'll investigate angular momentum in detail later. A preview of coming attractions:

- Just as moment of inertia is the rotational analog to mass, and torque is the rotational analog to force, *angular momentum* is the rotational analog to linear momentum.



- The angular momentum of a particle, ℓ, with respect to the origin O is: $\ell = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$.
- The product $\vec{r} \times \vec{p}$ is in a plane perpendicular to the plane containing ***r*** and ***p*** and in this case is out of the plane of the page.
- Angular momentum is a vector and it's direction is determined from the right hand rule. The magnitude of the angular momentum vector is ***rp***sinϕ.
- Notice that a particle does not have to rotate about O in order to have angular momentum with respect to O.
- Notice that just as Newton's second law may be written in terms of linear momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

  it may also be written in terms of angular momentum:

$$\vec{\Gamma} = \frac{d\vec{\ell}}{dt}$$

- Angular momentum may be written in terms of moment of inertia and angular velocity for a rigid body and a fixed axis:

$$\vec{\ell} = I\vec{\omega}$$

Translational and Rotational analogs

Newton's second law for translation/rotation: $\vec{F} = m\vec{a}$ $\vec{\Gamma} = I\vec{\alpha}$

Work in translating/rotating systems: $W = \int \vec{F} \cdot d\vec{s}$ $W = \int \Gamma d\theta$

Kinetic energy in translating/rotating systems: $KE = \frac{1}{2}mv^2$ $KE = \frac{1}{2}I\omega^2$

Power in translating/rotating systems: $P = Fv$ $P = \Gamma\omega$

Momentum in translating/rotating systems: $\vec{p} = m\vec{v}$ $\vec{\ell} = I\vec{\omega}$

Example 1. The drive shaft of an automobile rotates at 3600 rpm and produces 80 bhp. Compute the torque developed by the drive train.

$$\omega = \frac{3600\text{rev}}{\text{min}} \times \frac{2\pi\text{rad}}{\text{rev}} \times \frac{1\text{min}}{60\text{s}} = 120\pi\text{rad}\cdot\text{s}^{-1}$$

$$P = 80\text{bhp} \times \frac{746\,\text{watts}}{\text{bhp}} = 59700\text{watts}$$

$$\Gamma = \frac{P}{\omega} = \frac{59700\text{watts}}{120\pi\text{rad}\cdot\text{s}^{-1}} = 158\text{N}\cdot\text{m}$$

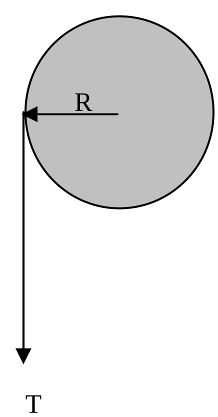
Example 2. A force, $\vec{F} = 2\hat{i} + 3\hat{j} - \hat{k}$, is applied to a particle located at $\vec{r} = \hat{i} - 3\hat{k}$. Compute the torque on this particle.

$\Gamma = \vec{r} \times \vec{F}$ (Definition of torque)

$$\Gamma = \begin{vmatrix} i & j & k & i & j \\ 1 & 0 & -3 & 1 & 0 \\ 2 & 3 & -1 & 2 & 3 \end{vmatrix} = \left(0\hat{i} - 6\hat{j} + 3\hat{k}\right) - \left(0\hat{k} - 9\hat{i} - 1\hat{j}\right) = 9\hat{i} - 5\hat{j} + 3\hat{k}$$

Example 3. A rope is wrapped several times around a uniform cylinder of radius = 0.1 meter (for a uniform cylinder $I = 1/2MR^2$), mass = 50 kg, that rotates about its axis of cylindrical symmetry. Find α if the rope is pulled with a force of 20 N.

$$\Gamma = I\alpha \therefore \frac{\Gamma}{I} = \alpha = 8s^{-2}$$



Example 4. An electric motor exerts a constant torque of 10 N · m on a grindstone shaft. The moment of inertia of the shaft is 2 kg · m^2. If the shaft starts from rest compute the work done in 8 seconds by the motor and the kinetic energy of the shaft at *t* = 8 seconds. Find the average power exerted during this interval.

$$\alpha = \frac{\Gamma}{I} = \frac{10N \cdot m}{2kg \cdot m^2} = 5s^{-2}$$

$$\omega_0 = 0$$

$$\omega_{t=8} = \alpha t = \left(5s^{-2}\right)\left(8s\right) = 40s^{-1}$$

$$KE = \frac{1}{2} I\omega^2 = 1600J$$

$$\theta = \frac{1}{2}\alpha t^2 = 160rad$$

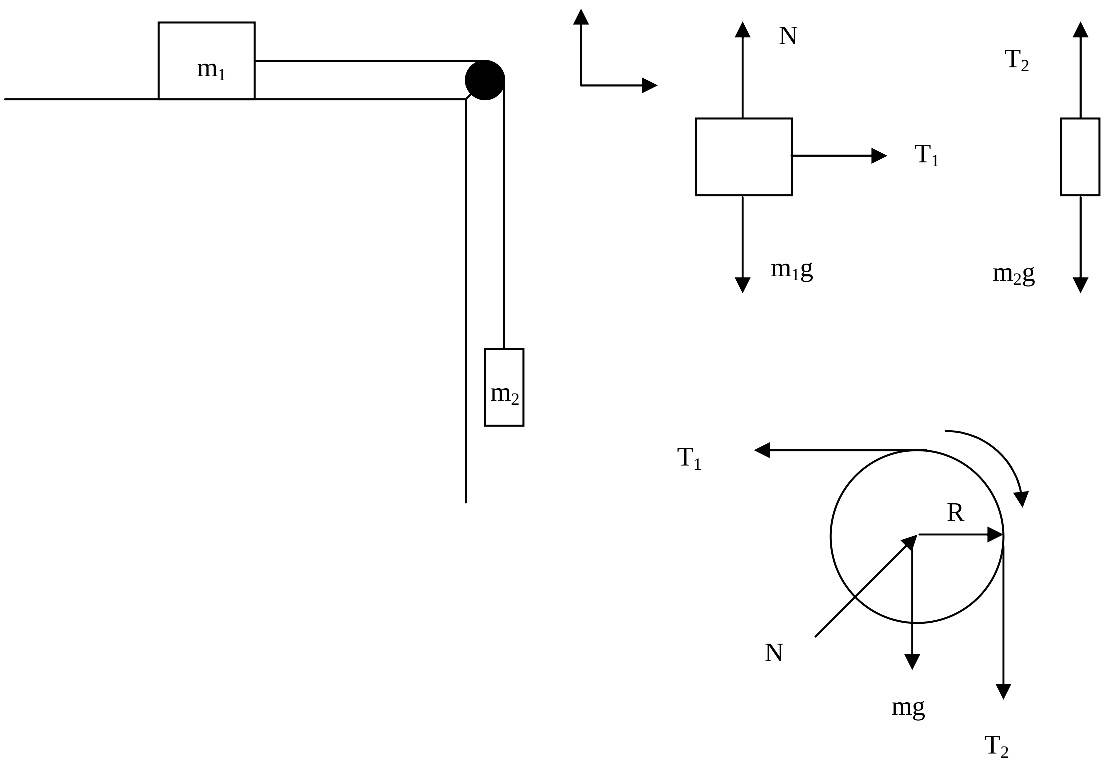
$$\text{Work} = \Gamma\theta = 1600J$$

Could also use the definition:

$$W = \int \Gamma d\theta = \int Pdt = \int \Gamma \omega dt = \int \Gamma\left(\alpha t\right)dt = \left(10N \cdot m\right)\left(5s^{-2}\right)\int_{t=0}^{t=8} tdt = \left(25kg \cdot m^2 \cdot s^{-4}\right)t^2 = 1600J$$

average power = $\frac{1600J}{8s} = 200\text{watts}$

Example 5. Find T_1, T_2, and a for the following system. The tabletop surface is smooth. The pulley has mass M and radius R. $I = MR^2$



Since this is a real pulley (it does more than merely redirect the direction of a force in space) the tension in the cord that connects the masses is not the same everywhere. It is the difference between T_1 and T_2 that provides the torque that causes the pulley to turn.

Recalling that: $\Gamma = I\alpha$, from the FBD's above:

- $T_1 = m_1 a_1$
- $T_2 - m_2 g = -m_2 a_2$
- $T_2 R - T_1 R = I\alpha = (MR^2)\alpha$

Note: $a_1 = a_2 = a_t = R\alpha$

1. $T_1 = m_1 a$
2. $T_2 - m_2 g = -m_2 a$
3. $T_2 - T_1 = Ma$

Three equations and three unknowns

Multiply equation #2 by -1 and add all three together:

$$T_1 - T_2 + T_2 - T_1 + m_2 g = m_1 a + m_2 a + Ma$$

$$m_2 g = m_1 a + m_2 a + Ma$$

Solving for acceleration:

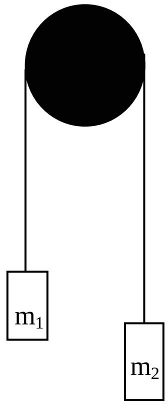
$$a = \frac{m_2}{m_1 + m_2 + M} g$$

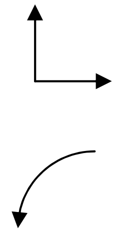
Now solving for T:

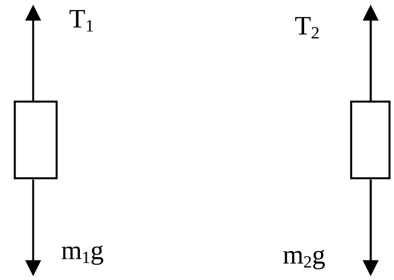
$$T_1 = \frac{m_1 m_2}{m_1 + m_2 + M} g$$

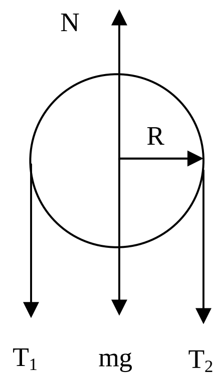
$$T_2 = \frac{(m_1 + M) m_2}{m_1 + m_2 + M} g$$

Example 6. Find the acceleration of the following system ($m_1 > m_2$). The pulley has mass M, radius R, and $I=MR^2$. Note $T_1 \neq T_2$.









From the FBD's

- $T_1 - m_1 g = -m_1 a_1$
- $T_2 - m_2 g = m_2 a_2$
- $T_1 R - T_2 R = I\alpha = MR^2\alpha$

Note: N & mg exert no torque about the pivot.

Note: $a_1 = a_2 = a_t = R\alpha$

1. $T_1 - m_1 g = -m_1 a$
2. $T_2 - m_2 g = m_2 a$
3. $T_1 - T_2 = Ma$

Multiply equation #1 by -1 and add all three together:

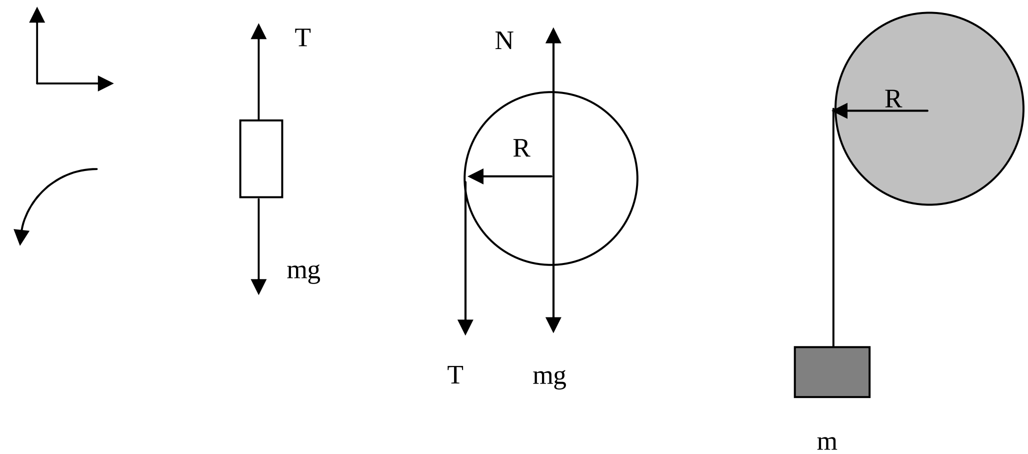
$$-T_1 + m_1 g + T_2 - m_2 g + T_1 - T_2 = m_1 a + m_2 a + Ma$$

$$m_1 g - m_2 g = m_1 a + m_2 a + Ma \therefore a = \frac{m_1 - m_2}{m_1 + m_2 + M} g$$

Find the velocity of the blocks after m_1 has fallen a distance h.

From kinematics: $v = \sqrt{2ah} = \sqrt{2\frac{m_1 - m_2}{m_1 + m_2 + M}gh} = \sqrt{2\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}gh}$

Example 7. A rope is wrapped several times around a uniform cylinder of radius *R* (for a uniform cylinder $I = 1/2MR^2$), mass *M*, setup so that it rotates about its axis of cylindrical symmetry. The torque is supplied by a hanging mass, *m*, as shown below. Determine the angular acceleration of the system and the tension in the cord.



(1) $T - mg = -ma$ (Note: $T \neq mg$)

(2) $\Gamma = TR = I\alpha = \left(\frac{1}{2}MR^2\right)\alpha$

Note: $a_t = R\alpha = a_{mass} = a$

(3) $\Rightarrow T = \frac{1}{2}Ma$

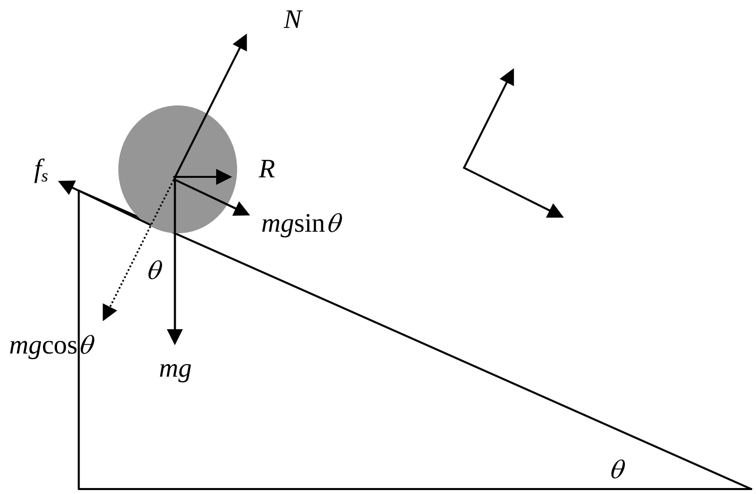
Combining equations (1) & (3):

$$mg - ma = \frac{1}{2}Ma \therefore mg = \left(m + \frac{M}{2}\right)a$$

$$a = \frac{mg}{m + \frac{M}{2}} = \frac{g}{1 + \frac{M}{2m}}$$

$$T = mg - ma = mg - m\left(\frac{g}{1 + \frac{M}{2m}}\right) \rightarrow T = \frac{mg}{1 + \frac{2m}{M}}$$

Example 8. A uniform cylinder rolls down a ramp inclined at an angle of θ to the horizontal. What is the linear acceleration of the cylinder at the bottom of the ramp?



$$\sum F_x = mg\sin\theta - f_s = ma_{cm} \qquad (1)$$

$$\sum \Gamma = Rf_s = I\alpha \qquad (2)$$

Note: $I\alpha = \frac{1}{2}mR^2\alpha$

Note: $R\alpha = a_{cm}$

$$f_s = \frac{mRa_{cm}}{2R} = \frac{1}{2}ma_{cm}$$

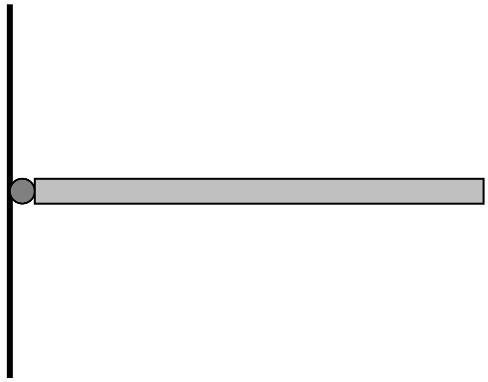
(2) → (1)

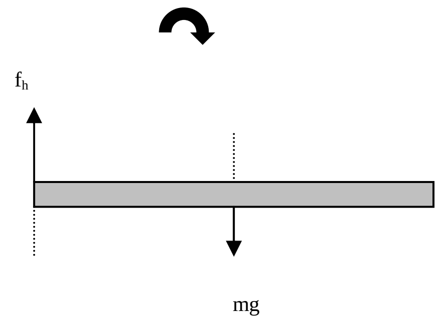
$$\sum F_x = mg\sin\theta - \frac{1}{2}ma_{cm} = ma_{cm}$$

$$g\sin\theta = \frac{3}{2}a_{cm}$$

$$a_{cm} = \frac{2g\sin\theta}{3}$$

Example 9. A rod of length ℓ and mass *m* is affixed to a vertical wall by a hinge on its left side as shown below. If the right side of the rod is released and the rod is allowed to swing down into the wall find the angular acceleration of the rod.





The torque to rotate this rod is supplied to gravity. Remember that gravity acts at the COM.

$I = \frac{1}{3}m\ell^2$ (moment of inertia of the rod about the hinge)

$\Gamma = I\alpha = mg\frac{\ell}{2}$ (torque = force x distance)

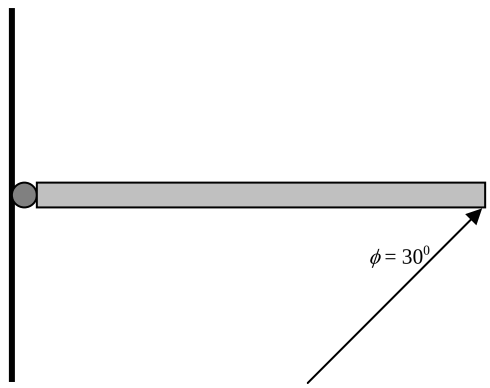
$$mg\frac{\ell}{2} = \frac{1}{3}m\ell^2\alpha$$

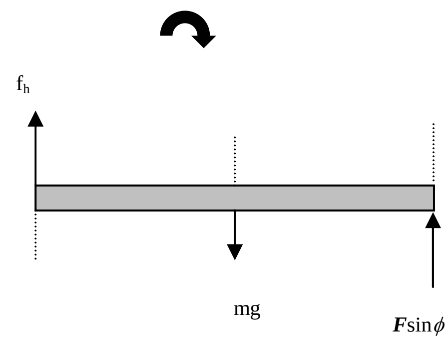
$$\alpha = \frac{3g}{2\ell}$$

What is the tangential acceleration of the free end of the rod?

$$a_t = \ell\alpha \therefore a_t = \frac{3}{2}g$$

Example 10. A rod of mass *m* and length ℓ is affixed to a vertical wall by a hinge on its left side as shown below. A force *F* is applied as shown. Compute the net torque on the system and the angular acceleration of the system.





$$I = \frac{1}{3}m\ell^2$$

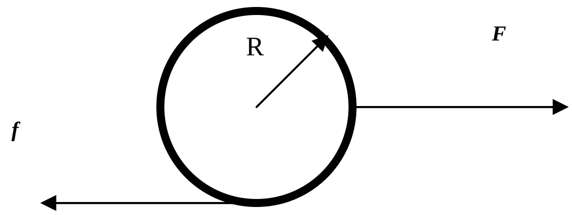
$$\Sigma\Gamma = mg\frac{\ell}{2} - F\sin\phi\ell = I\alpha$$

$$mg\frac{\ell}{2} - F\sin\phi\ell = \frac{1}{3}m\ell^2\alpha$$

$$\frac{1}{2}mg - F\sin\phi = \frac{1}{3}m\ell\alpha$$

$$\alpha = \frac{3g}{2\ell} - \frac{3F\sin\phi}{m\ell}$$

Example 11. *A* constant horizontal force ***F*** is applied to a lawn roller in the form of a uniform solid cylinder of radius *R* and mass *M*. If the roller is to roll without slipping, show that the acceleration of the center of mass is $a_{cm} = \frac{2F}{3M}$, and that the coefficient of static friction between the roller and the ground is $\mu_s = \frac{F}{3Mg}$.



From the FBD above generate equations that sum both forces and torques:

$$\sum F = F - f = Ma_{cm} \qquad (1)$$

$$\sum \Gamma = fR = I\alpha \qquad (2)$$

Note: $I\alpha = \frac{1}{2}MR^2\alpha$

Note: $R\alpha = a_{cm}$

The acceleration of the COM:

$$f = \frac{I\alpha}{R} = \frac{1}{2}\frac{MR^2\alpha}{R} = \frac{MRa_{cm}}{2R} = \frac{1}{2}Ma_{cm} \qquad (2)$$

$$F - \frac{1}{2}Ma_{cm} = Ma_{cm} \qquad (2) \rightarrow (1)$$

$$F = \frac{3}{2}Ma_{cm} \therefore a_{cm} = \frac{2F}{3M} \qquad \text{(QED)}$$

The coefficient of static friction:

$$f = \mu_s Mg$$

$$fR = I\alpha \rightarrow \mu_s MgR = \frac{1}{2}MR^2\alpha$$

$$\mu_s g = \frac{1}{2}a_{cm} \therefore \mu_s = \frac{1}{2}\frac{2F}{3Mg} = \frac{F}{3Mg} \qquad \text{(QED)}$$

Example 12. A bowling ball with initial velocity v_0 slides without rolling down a lane. The coefficient of static friction between the ball and lane is μ. Show that pure rolling motion occurs when the velocity of the center of mass of the ball is $\frac{5}{7}v_0$ and the distance traveled is $\frac{12}{49}\frac{v_0^{\ 2}}{\mu g}$.

Note: $v_{cm} = R\omega$ for pure rolling motion.

Rolling motion occurs due to a torque on the ball imparted by the force of static friction between the ball and the floor. The force of friction is $f = ma_{cm} = \mu mg \therefore a_{cm} = \mu g$. Since the ball is slowing down, $a_{cm} = -\mu g$. The torque exerted by this force is $R\mu mg = Rf$.

$$v = v_0 + at$$

(1) $$v = v_0 - \mu gt$$

$$\Gamma = I\alpha$$
$$\Gamma t = I\alpha t$$
$$\Gamma t = I\omega$$

(2) $$Rft = I\omega$$

(3) $$\omega = \frac{v}{R} \rightarrow \frac{Rft}{I} = \frac{v_0 - \mu gt}{R}$$

$$\frac{R\mu mgt}{\frac{2}{5}mR^2} = \frac{v_0 - \mu gt}{R}$$

$$\frac{\mu gt}{\frac{2}{5}} = v_0 - \mu gt$$

$$\frac{5}{2}\mu gt + \mu gt = v_0$$

$$v_0 = \frac{7}{2}\mu gt$$

(4) $$t = \frac{2v_0}{7\mu g}$$

(4) → (1) $$v = v_0 - \mu g \frac{2v_0}{7\mu g} \therefore v = \frac{5}{7} v_0$$ QED

The distance traveled from $t = 0$ to $t = t$ is: $x = \int_0^t v dt = \int_0^t (v_0 - \mu g t) dt$

$$x = \left(v_0 t - \mu g \frac{t^2}{2} \right) \Big|_0^t$$

$$x = v_0 \frac{2}{7} \frac{v_0}{\mu g} - \frac{\mu g}{2} \left(\frac{2}{7} \frac{v_0}{\mu g} \right)^2$$

$$x = \frac{2}{7} \frac{{v_0}^2}{\mu g} - \frac{2}{49} \frac{{v_0}^2}{\mu g}$$

$$x = \frac{12}{49} \frac{{v_0}^2}{\mu g}$$