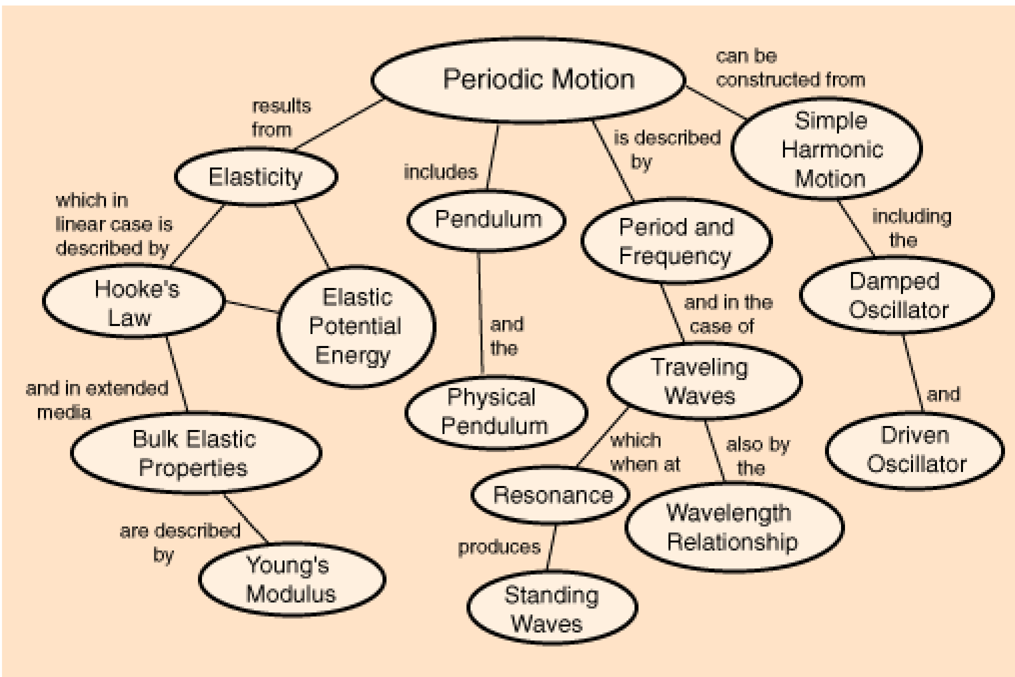
Simple Harmonic Motion

Harmonic Oscillators

- Obey Hooke's Law $\vec{F} = -k\vec{x}$
- Simple harmonic oscillators are free-running and conservative, i.e., no resistive forces are present.
- Light, sound, suspension, uniform circular motion all quantum phenomena are governed by harmonic motion
- Periodic motion with additional constraints



Courtesy of Rod Nave, Georgia State University
http://hyperphysics.phy-astr.gsu.edu/hbase/permot.html#permot

Also check out:

http://www.kettering.edu/~drussell/Demos/SHO/mass.html
http://monet.physik.unibas.ch/~elmer/pendulum/harmosc.htm
http://webphysics.davidson.edu/Applets/TaiwanUniv/springWave/springWave.html
http://www.kettering.edu/~drussell/Demos/SHO/mass-force.html
http://www.physics.nmt.edu/~raymond/classes/ph13xbook/node117.html
http://www.stedwards.edu/science/dnaples/de_html/chapter_4/sec_4_1/section_411.htm

SHO equations:

$$F = -kx$$
$$ma = -kx$$
$$m\frac{d^2x}{dt^2} = -kx$$

The latter equation is often written: $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

An alternative form of Hooke's Law: $$m\ddot{x} + kx = 0$$

- Physically simple harmonic motion is very similar to uniform circular motion and it is easy to show that displacement, velocity and acceleration vary as sines and cosines.
- Mathematically it may be shown that there are various families of solutions to the equations above, some involving sines and cosines and other involving other functions. Among them:

$x(t) = A\cos(\omega t)$ or $x(t) = A\cos(\theta)$ (1)

$x(t) = A\sin(\omega t)$ (2)

$x(t) = A\cos(\omega t) + B\sin(\omega t)$ (a linear combination of 1 and 2) (3)

$x(t) = A\cos(\omega t + \alpha)$ (α is a phase angle) (4)

$x(t) = A\sin(\omega t + \alpha)$ (5)

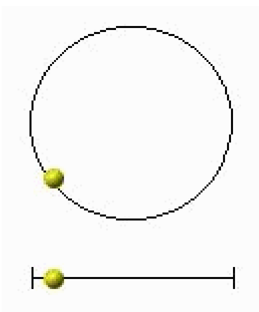
$x(t) = A\cos(\omega t + \alpha) + B\sin(\omega t + \alpha)$ (6)

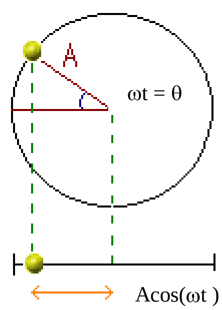
where ***A*** and ***B*** are constants which must be determined. Since all of these solutions yield displacement as a function of time it follows that:

$$v(t) = \dot{x}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$
$$a(t) = \ddot{x}(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

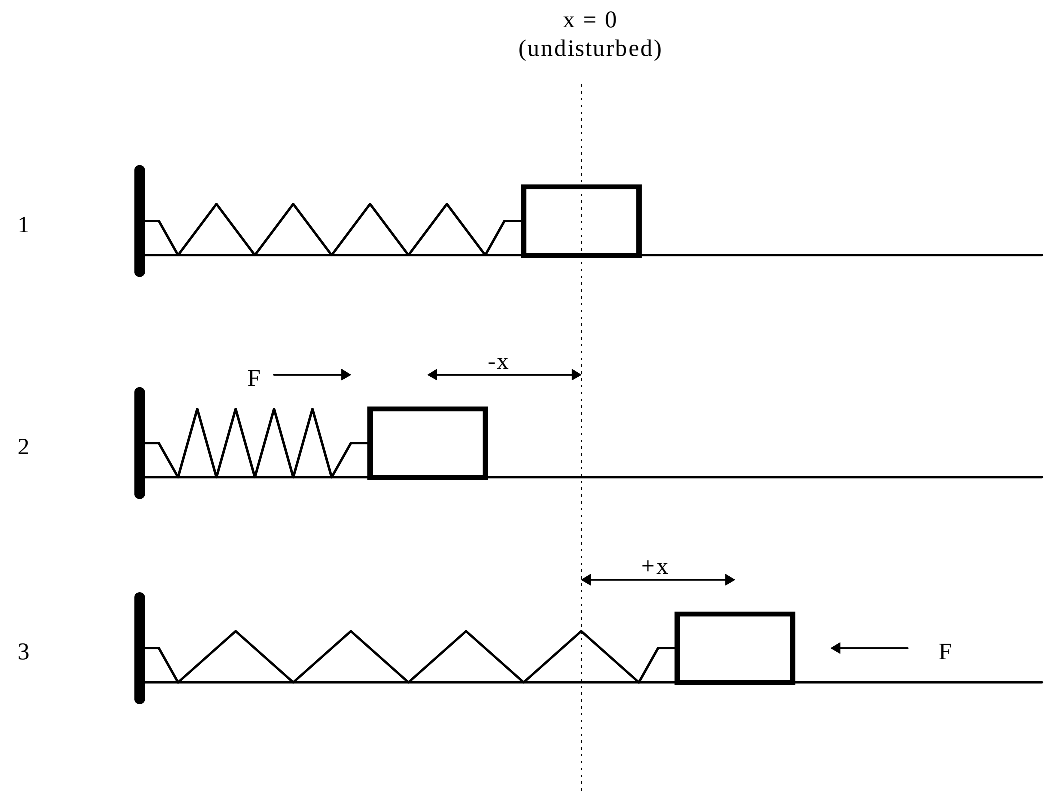
A and ***B*** are components of the amplitude or maximum displacement of the SHO. Sometimes these constants are referred to as C_1 and C_2.

http://hypertextbook.com/chaos/41.shtml
Simple Harmonic motion is analogous to uniform circular motion and this is useful in visualizing a solution to position, velocity and acceleration as functions of time.





- Since both particles have the same motion in the horizontal direction the circular functions (sin and cos) may be used to describe the displacement of the particle in terms of some maximum displacement (A) with respect to an equilibrium position.
- The projection of A down onto the horizontal line varies in length as $x = A\cos(\omega t)$



- When the particle passes through $x = 0$, $F = 0$, $a = 0$, and $v = v_{max}$
- When the particle passes through $x = \pm A$, $F = F_{max}$, $a = a_{max}$, $v = 0$ (How can velocity be zero when acceleration is at a maximum value?).
- Linear frequency, f, is the number of oscillations per second of a free-running system
- Angular frequency, $\omega = 2\pi f = \sqrt{\frac{k}{m}}$
- The period, T, is the amount of time it takes for one full cycle of oscillation.
- The natural frequency of a SHO is the frequency that for a given mass/spring combination oscillates, $\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{elasticity}{inertia}}$
- a_{max} occurs at an angular displacement of π rad and v_{max} occurs at an angular displacement of $\pi/2$ rad

Example 1 A guitar string oscillates harmonically with a frequency of 512 Hz (the *C* note above middle *C*). If the amplitude of oscillation of the center point of the string is 0.002m, find the maximum velocity of the string and the maximum acceleration of the string.

In order to find the desired values we must first solve for the constants *A* and *B* using initial conditions. Even though it's possible to do this intuitively (the amplitude of oscillation is given) let's show that the amplitude is all represented by one constant.

At time t = 0: f = 512 Hz, x = 0.002m, v = 0

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \therefore 0.002 = A\cos(0) + B\sin(0) \therefore 0.002 = A(1) + B(0)$$

so ***A* = 0.002m**

Now let's see what the value of ***B*** is:

$$v(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \therefore 0 = -\omega 0.002\sin(0) + \omega B\cos(0) \therefore 0 = \omega B$$

and since $\omega \neq 0$, ***B* = 0m**

So we've verified that A is the amplitude and B equals zero.

Note: $\omega = 2\pi f = 1024\pi$ rad/s

$$v_{\max}(t) = -\omega A\sin\left(\frac{\pi}{2}\right) + \omega B\cos\left(\frac{\pi}{2}\right)$$

$$v_{\max}(t) = -(1024\pi)(0.002m)\sin\left(\frac{\pi}{2}\right) = 6.43m/s$$

Finally:

$$a(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$
$$a_{\max}(t) = -(1024\pi)^2(0.002m)\cos(\pi) - 0$$
$$a_{\max}(t) = 2.07\times 10^4 m/s^2$$

Example 2 A piston executes SHM with an amplitude of 0.1 meters. If it passes through the center of motion with a velocity of 0.5 m/s, find the period, *T*.

At time t = 0: x = 0. 1m, v = 0

$x(t) = A\cos(\omega t) + B\sin(\omega t) \therefore 0.1 = A\cos(0) + B\sin(0) \therefore 0.1 = A(1) + B(0)$ so ***A* = 0.1m**

By the same method as in Example 1, it may be shown that **B = 0**

v @ $\omega t = \pi/2$ = 0.5 m/s.

$$v(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \therefore 0.5 = -\omega 0.1\sin\left(\frac{\pi}{2}\right) + 0 \therefore 0.5 = -\omega(0.1) \therefore \omega = 5rad/s$$

Note: the minus sign was dropped, why?

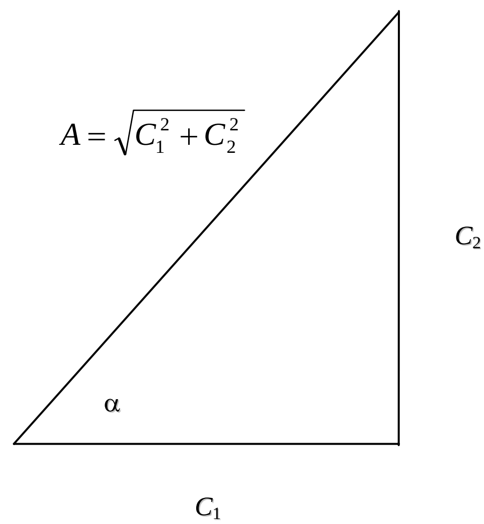
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5rad/s} = 1.26s$$

Example 3 Recall that C_1 and C_2 (*A* and *B* in the previous examples) are components to the instantaneous amplitude for a SHO in the solution equations used in the previous examples. What if C_1 and C_2 are both non-zero?

$$x(t) = A\sin(\omega t + \alpha)$$
$$x(t) = A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$
$$x(t) = A\cos\alpha\sin\omega t + A\sin\alpha\cos\omega t$$

$$\sin\alpha = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{A}$$
$$\cos\alpha = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{A}$$

$$A\sin\alpha = C_2$$
$$A\cos\alpha = C_1$$



$$\therefore\ x(t) = A\cos(\omega t) + B\sin(\omega t)$$

So: $$x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$x(t) = A\cos(\omega t + \alpha) + B\sin(\omega t + \alpha)$$
$$x(t) = A\sin(\omega t + \alpha)$$

Are all equally valid (and in fact the same) solutions to the SHO equation derived from Hooke's Law where amplitude, $A = \sqrt{C_1^2 + C_2^2}$.

Could you have predicted this result given the orthogonal relationship between the sine and cosine functions?

Example 4 A system executes SHM with an angular frequency of 5 rad/s. If it passes through the equilibrium position with a velocity of 0.5 m/s, find the amplitude of the system

At time *t* = 0: *x* = 0. 0m, *v* = 0.5m/s, ω = 5s^{-1}

$x(t) = A\cos(\omega t) + B\sin(\omega t) \therefore 0.0 = A\cos(0) + B\sin(0) \therefore 0.0 = A(1) + B(0)$ so ***A* = 0.0m**

v @ ωt = 0 (because θ = 0 at the equilibrium position which is where the system is at t = 0) = 0.5m/s

$$v(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \therefore 0.5 = 0 + \omega B\cos(0) \therefore 0.5 = (5s^{-1})B(1) \therefore B = 0.1m$$

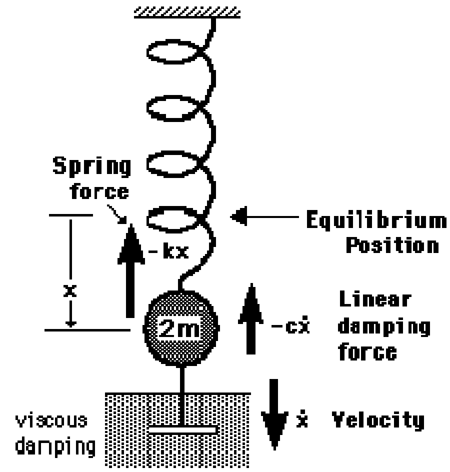
So the amplitude of the system is 0.1 meters

Damped Harmonic Oscillators

A damped harmonic oscillator does not run free. The amplitude of each successive oscillation is less that the one preceding it. Such a harmonic system is described by the equation:

$$m\ddot{x} + c\dot{x} + kx = 0$$

where *c* is a *damping factor* that controls the decay of successive oscillations.



The motion equation of the form

$$m\ddot{x} + c\dot{x} + kx = 0$$

may be solved in the form $\mathbf{x} = \mathbf{e^{qt}}$

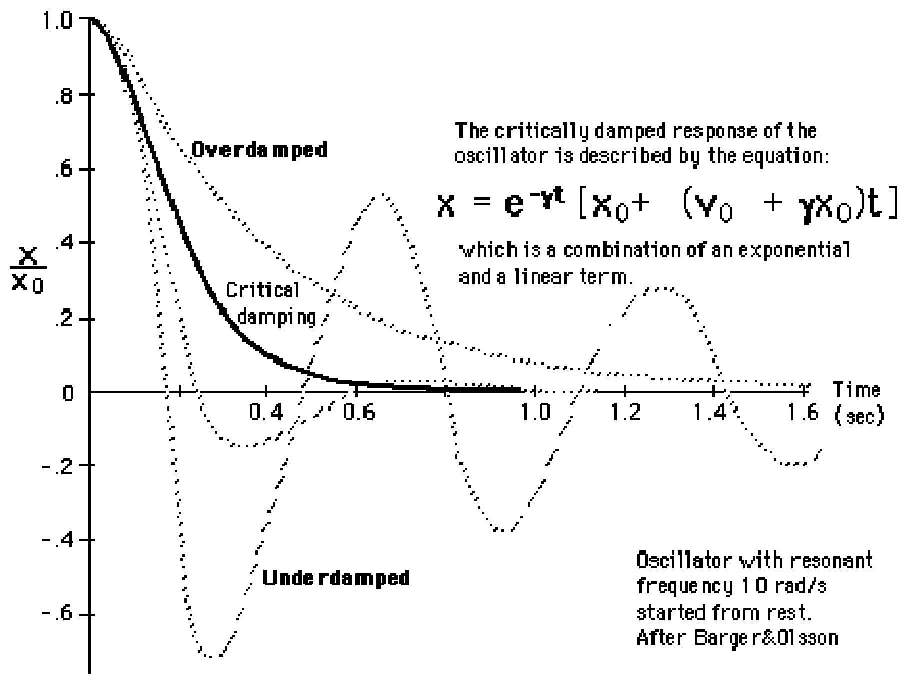
$$mq^2 + cq + k = 0$$

The roots of the auxiliary equation are

$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

which give the three cases:

- underdamped
- overdamped
- critically damped



Courtesy of Rod Nave, Georgia State University
http://hyperphysics.phy-astr.gsu.edu/hbase/oscda.html

Three possible values for the discriminant in the solution quadratic for a DHO:

I $c^2 - 4mk > 0$ overdamped
II $c^2 - 4mk = 0$ critical damping
III $c^2 - 4mk < 0$ underdamped

For a damped oscillator, it may be shown that a solution to:

$$m\ddot{x} + c\dot{x} + kx = 0$$
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$
$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

is: $x(t) = e^{-\lambda t}\left(C_1 + C_2 t\right)$

where $\frac{c}{m} = 2\lambda$ and $\frac{k}{m} = \omega^2 = \lambda^2$

$$x(t) = e^{-\lambda t}\left(C_1 + C_2 t\right)$$
$$v(t) = \dot{x}(t) = -\lambda e^{-\lambda t}(C_1 + C_2 t) + C_2 e^{-\lambda t}$$
$$a(t) = \ddot{x}(t) = \lambda^2 e^{-\lambda t}(C_1 + C_2 t) - C_2 \lambda e^{-\lambda t} - C_2 \lambda e^{-\lambda t}$$
$$a(t) = \lambda^2 e^{-\lambda t}(C_1 + C_2 t) - 2\lambda C_2 e^{-\lambda t}$$

Now substitute into: $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \lambda^2 x = 0$

$$2\lambda^2 e^{-\lambda t}(C_1 + C_2 t) - 2\lambda C_2 e^{-\lambda t} + 2\lambda^2 e^{-\lambda t}(C_1 + C_2 t) + 2\lambda C_2 e^{-\lambda t} = 0$$
$$0 = 0$$
$$Q.E.D.$$

Example 5 A motorcycle suspension is critically damped. The period of free oscillation for the system (when the damping is turned off) is 1 second. If the wheel is displaced from equilibrium 5 cm, and the damped system is allowed to return the wheel to its equilibrium position (starting with a velocity of zero) find its displacement from equilibrium after one second.

In the case of a critically damped oscillator the two roots to the auxiliary equation are equal to each other. It may be easily shown that: $q_1 = q_2 = \frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_0 = \frac{2\pi}{T_0}$.

Hence: $\frac{c}{2m} = 2\pi s^{-1}$ for a period (T_0) of 1 second. Now recall that for a damped oscillator displacement as a function of time may be expressed as:

$$x(t) = e^{-\lambda t}(C_1 + C_2 t)$$

where $\lambda = \sqrt{\frac{k}{m}} = \frac{c}{2m} = 2\pi s^{-1}$.

We need to find C_1 and C_2. To do this we will evaluate the boundary conditions. For t = 0:

$$x(0) = e^{-0}(C_1 + C_2(0))$$

which, when evaluated yields:

$$x(0) = (C_1)$$

or C_1 = 0.05m

To find C_2 we differentiate:

$$\dot{x}(t) = v(t) = (C_2 - \lambda C_1 - \lambda C_2 t)e^{-\lambda t}$$

which, when evaluated at t = 0 (when the velocity is also 0), yields:

$$v(0) = (C_2 - \lambda C_1) = 0$$

so that $C_2 = \lambda C_1 = \lambda(5cm)$

Putting all of this together at time t = 1 yields:

$$x(t) = e^{-2\pi s^{-1}t}\left(5cm + \left(2\pi s^{-1}(5cm)\right)t\right)$$

$$x(t) = 5cm(1 + 2\pi s^{-1}t)e^{-2\pi s^{-1}t}$$

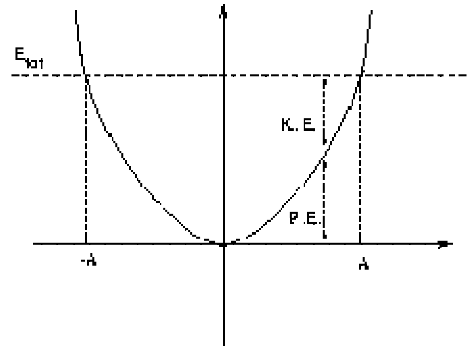
$$x(1) = 5cm(0.0136) = 0.068cm$$

The system has essentially returned to equilibrium.

Energy of a SHO

Following the SHM assumptions, PE and KE are conserved in a simple harmonic oscillator and

E = PE +KE or *E = K + U*



Since

$$KE = \frac{1}{2}mv^2$$

$$PE - \frac{1}{2}kx^2$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\omega^2 A^2 \sin^2(\omega t + \alpha)\right)$$

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}k\left(A^2 \cos^2(\omega t + \alpha)\right)$$

Because the squares are always positive, K and U are always positive.

Total Energy: ***E = K + U***

$$E = \frac{1}{2}m\left(\omega^2 A^2 \sin^2(\omega t + \alpha)\right) + \frac{1}{2}k\left(A^2 \cos^2(\omega t + \alpha)\right)$$

Note: $\omega^2 = \frac{k}{m}$

$$E = \frac{1}{2}m\left(\frac{k}{m}A^2 \sin^2(\omega t + \alpha)\right) + \frac{1}{2}k\left(A^2 \cos^2(\omega t + \alpha)\right)$$

$$E = \frac{1}{2}kA^2 \sin^2(\omega t + \alpha) + \frac{1}{2}kA^2 \cos^2(\omega t + \alpha)$$

$$E = \frac{1}{2}kA^2(\sin^2(\omega t + \alpha) + \cos^2(\omega t + \alpha))$$

Note: $\sin^2\alpha + \cos^2\alpha = 1$

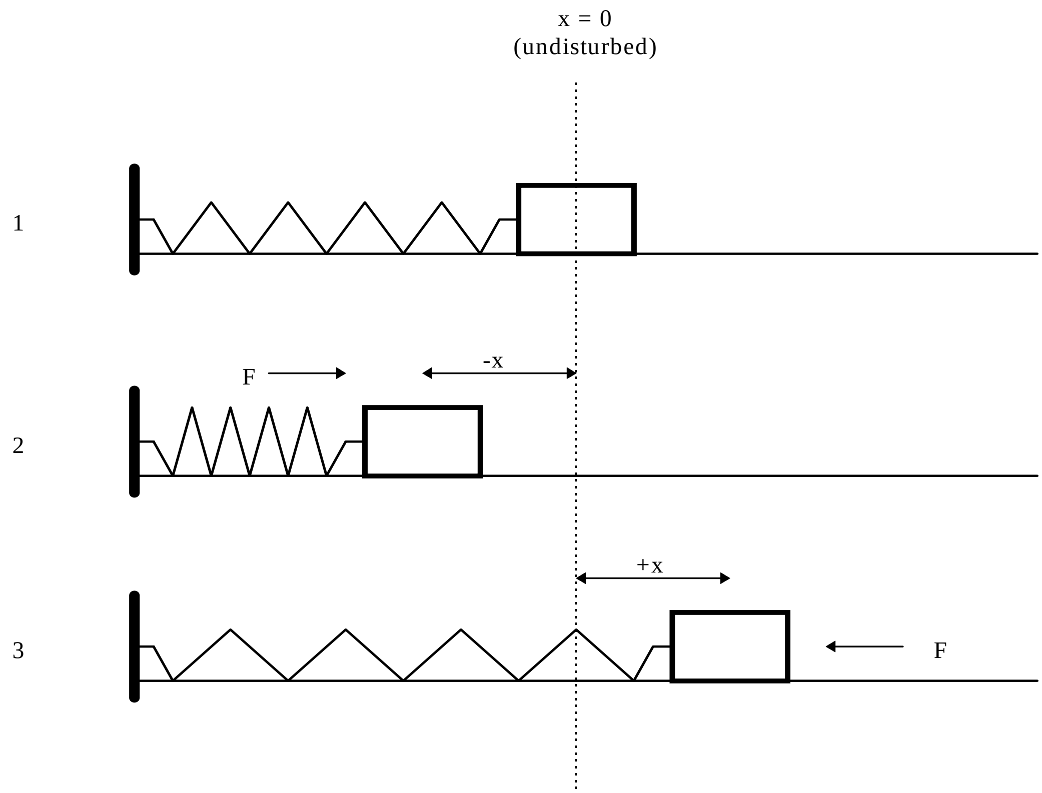
$$E = \frac{1}{2}kA^2$$

Total energy is constant and is continuously being converted from kinetic to potential and vice versa.

Resonance in Harmonic Systems – Driven Oscillators

So far we have look at two types of SHM, a free running SHO and a Damped HO. A third case of harmonic motion bears examination even at the introductory level, that of a forced oscillator.

Consider the mass/spring system illustrated below.



- What if, instead of allowing this system to oscillate undisturbed at its natural frequency, ω_0, it was periodically disturbed by a force acting from the right.
- We could conjure up a perturbing force that acts as a periodic displacement of the mass by the force by a sinusoidal factor of $d = d_0(\cos\omega_F t)$
- Other non-harmonic driving forces are possible but are much more complex to examine requiring *Fourier Series* analysis.
- This force could have a period, ω_f, greater than, less than, or equal to the natural frequency of the oscillator.
- The behavior of the driven harmonic system varies with each of these cases.

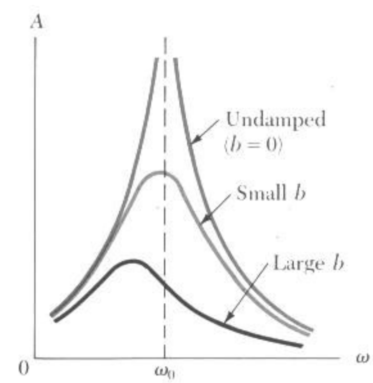
DE's for forced harmonic motion

$$m\ddot{x} + kx = f_0 \cos(\omega t) \qquad (1)$$
$$m\ddot{x} + \beta\dot{x} + kx = f_0 \cos(\omega t) \qquad (2)$$
$$m\ddot{x} + \beta\dot{x} + kx = f_0 e^{i\omega t} \qquad (3)$$

The steady state solution for (2) is: $x(t) = A\cos(\omega t + \alpha)$ where $A = \dfrac{\dfrac{F_0}{m}}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{\beta\omega}{m}\right)^2}}$

Here $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the frequency of the undamped oscillator (β = 0). Can you show that this is a valid solution?

- In the case of damping, the damped oscillator being driven by an external force does not act like a normal damped oscillator since the driving force has enough energy to overcome the energy lost to damping.
- The system oscillates at the driving frequency, ω. For small values of β the amplitude becomes very large when ω approaches ω_0. This condition is knows as *resonance* with ω_0 being the *resonant frequency* of the system.
- At resonance energy from the driving force is being transferred into the system under the most favorable conditions
- The shape of the resonance curve depends on the size of β
- The *sharpness* of the resonance peak is known as the *quality factor*



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