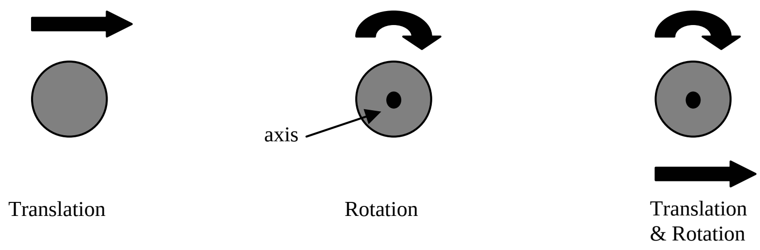
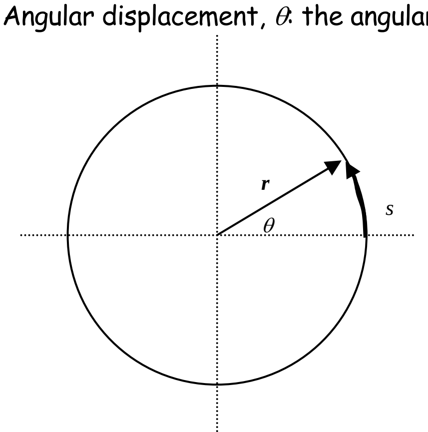
Rotation/Rotational Inertia/Angular Momentum

Translation vs. Rotation



Rotational variables

Angular displacement, θ: the angular distance from some fixed reference.



$$\theta = \frac{s}{r} \text{ radians}$$

We will assume that angular displacement is + if it is counterclockwise from the $+x$ axis. Although it has both magnitude and direction, angular displacement is not generally considered a vector quantity because addition of angular displacements is not communicative. Only in the limiting case of $d\theta$ can an angular displacement be considered a vector. Normally we are interested in angular displacement as a function of time or $\theta(t)$.

1 revolution = 360^0 = 2π radians

1 radian = 57.3^0 = 0.159 revolutions

A complete revolution is some multiple of 2π radians, e.g., 2π, 4π, 6π, etc.

Angular velocity, ω: the time rate of change of angular displacement. Angular velocity has the same value for all particles in a rotating system. Tangential velocity, which depends upon distance from the rotational axis, varies depending upon radius.

$$\vec{\omega}(t) = \frac{\Delta\vec{\theta}}{\Delta t} \quad \text{rad/sec}$$

Angular velocity is a *psuedovector*. The direction is determined from the *right hand rule*. If one curls their right hand around the axis of rotation with their fingers pointing in the direction of rotation, their thumb then gives the direction of the angular momentum vector. Note that the direction of the angular velocity vector is along the axis of rotation rather than in the direction of motion.

Angular acceleration, α: the time rate of change of angular displacement. Angular acceleration has the same value for all particles in a rotating system.

$$\vec{\alpha}(t) = \frac{\Delta\vec{\omega}}{\Delta t} \quad \text{rad/sec}^2$$

Angular acceleration is another psuedovector and its direction is also determined from the RHR.

Relating Linear and Angular variables

$s = \theta r$	radians	$\theta = \frac{s}{r}$
$v = r\omega$	radian measure	$\omega = \frac{v}{r}$
$a_t = \alpha r$	radian measure	$\alpha = \frac{a_t}{r}$
$a_r = \omega^2 r = \frac{v^2}{r}$	radian measure	

Linear/Rotational equations

$$v = v_0 + at \qquad \omega = \omega_0 + \alpha t$$

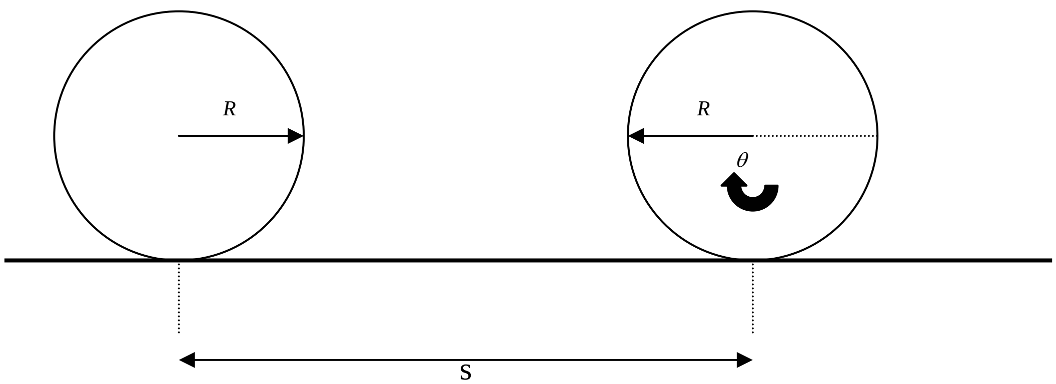
$$x - x_0 = v_0 t + \frac{1}{2}at^2 \qquad \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

General Rolling Motion

General rolling motion consists of both translation and rotation. Although analysis of the general rotary motion of a rigid body in space may be quite complicated, it is made easier by a few simplifying constraints. Initially we will consider only objects with an extremely high degree of symmetry about a rotational axis, e.g., hoops, cylinders, spheres.

Consider a *uniform* cylinder of radius *R* rolling on a rough (no slipping) horizontal surface.



As the cylinder rotates through an angular displacement θ, its center of mass (cm) moves through distance $s = r\theta$, or the same distance as the arc length.

$$s_{cm} = R\theta$$

$$v_{cm} = \frac{\Delta s}{\Delta t} = R\frac{\Delta \theta}{\Delta t} = R\omega$$

$$a_{cm} = \frac{\Delta v_{cm}}{\Delta t} = R\frac{\Delta \omega}{\Delta t} = R\alpha$$

If one looks at the velocity of a point on the surface of the cylinder in linear terms the situation is quite complicated. The total linear velocity is composed of two components: the tangential component, due strictly to rotation, and the translational component. It may be easily shown that the total linear velocity of a point at the very top of the cylinder, relative to the surface across which it rolls, is $2v_{cm}$ ($2R\omega$), and that the linear velocity of a point at the bottom of the cylinder (in contact with the surface) is zero, relative to the surface. The linear velocity of the axis around which the cylinder rotates is, of course, v_{cm}.

What does this imply about the nature of rolling motion vis a vie friction?

Rotational Kinetic Energy

Consider a rigid body as a system of particles rotating about a fixed axis in space with an angular velocity ω. Each particle has some kinetic energy: $KE_i = \frac{1}{2} m_i v_i^{\ 2}$.

Note that every particle has the same angular velocity, ω, but a different linear velocity v_i that depends upon its radial distance, r_i, from the axis of rotation, $v_i = r_i \omega$. Note: $\sum$ means "the sum of."

$$KE = \sum KE_i = \sum \frac{1}{2} m_i v_i^{\ 2} = \frac{1}{2} \sum m_i v_i^{\ 2} = \frac{1}{2} \sum m_i (r_i \omega_i)^2 = \frac{1}{2} \sum (m_i r_i^{\ 2}) \omega^2$$

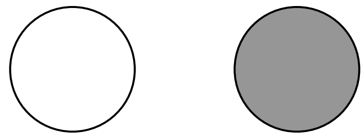
The quantity $m_i r_i^{\ 2}$ is known as the *moment of inertia*, *I*. Hence, rotational kinetic energy may be expressed:

$$KE = \frac{1}{2} I \omega^2 \text{ radians}$$

Moment of Inertia

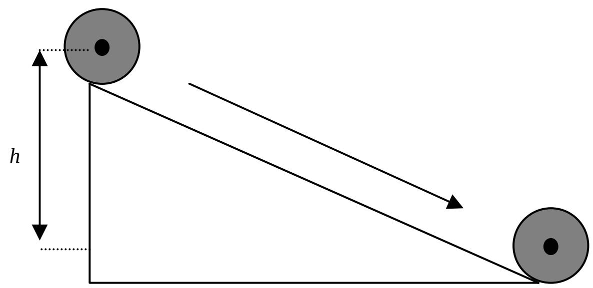
- $I = \sum m_i r_i$
- Rotational analog of mass
- Has units of kg · m^2
- Moments of Inertia are computed with respect to an axis
- Distribution of mass in a rotating body is significant

Demo – who wins the race?



A solid cylinder and a hollow cylinder of the same mass are rolled down an incline. Which gets to the bottom first? Do they arrive at the same time? What would happen if you dropped the two from the same height ignoring the resistance of air?

First consider a solid cylinder of radius *R* that rolls without slipping down an incline from some initial height *h*. The linear velocity of the cylinder at the bottom of the incline is v_{cm} and the angular velocity is ω.



If the cylinder starts from rest, all of its subsequent kinetic energy comes from gravitational potential energy. Because the cylinder is both translating and rotating as it moves down the plane, some of this initial energy goes into rotation and some goes into translation. This means that the linear velocity of the cylinder at the bottom of the plane is slower than it would be if the cylinder slid down the plane without rotating. Energy is still conserved, but the initial potential energy is now converted into two types of kinetic energy.

$$PE_i = KE_{f_{rot}} + KE_{f_{trans}}$$

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}{}^2$$

For pure rolling motion (i.e., no slipping) $v_{cm} = R\omega$.

$$mgh = \frac{1}{2}I\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

For a **solid cylinder** rotating about a symmetry axis down the length of the cylinder, $I = \frac{1}{2}MR^2$.

$$mgh = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}{}^2$$

$$gh = \frac{1}{2}\left(\frac{1}{2}R^2\right)\frac{v_{cm}{}^2}{R^2} + \frac{1}{2}v_{cm}{}^2 \rightarrow gh = \frac{1}{4}v_{cm}{}^2 + \frac{1}{2}v_{cm}{}^2 \therefore v_{cm} = \sqrt{\frac{4}{3}gh}$$

Note that this is less than $\sqrt{2gh}$ that would be the case for purely translational motion. Notice also that the result does not depend upon either the mass or radius of the cylinder.

We can also solve for angular velocity.

$$mgh = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + \frac{1}{2}m(R\omega)^2$$

$$gh = \frac{1}{2}\left(\frac{1}{2}R^2\right)\omega^2 + \frac{1}{2}(R\omega)^2 \rightarrow gh = \frac{1}{4}R^2\omega^2 + \frac{1}{2}R^2w^2 \therefore gh = \frac{3}{4}R^2\omega^2 \therefore \omega = \sqrt{\frac{4}{3}\frac{gh}{R^2}}$$

What is the ratio of rotational to translational energy?

$$\frac{KE_r}{KE_t} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}{}^2} = \frac{\frac{1}{4}}{\frac{1}{2}} = 50\%$$

What percentage of the total kinetic energy goes into rotation?

$$\frac{KE_r}{KE_{total}} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}{}^2 + \left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2} = \frac{\frac{1}{4}}{\frac{3}{4}} = 33\%$$

Now let's consider what happens when a thin hollow cylinder is released from rest side by side with a solid cylinder of identical mass and radius and both roll without slipping down an incline 1 meter in height.

a) What is the translational velocity of the hollow cylinder at the bottom of the incline?

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}^{\;2}$$

$$mgh = \frac{1}{2}I\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}^{\;2}$$

$$mgh = \frac{1}{2}\left(mR^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2}mv_{cm}^{\;2}$$

$$gh = \frac{1}{2}v_{cm}^{\;2} + \frac{1}{2}v_{cm}^{\;2} \therefore v_{cm} = \sqrt{gh} = 9.9m \cdot s^{-1}$$

b) What is the translational velocity of the solid cylinder at the bottom of the incline?

By the same reasoning $v_{cm} = \sqrt{\frac{4}{3}gh} = 11.4m \cdot s^{-1}$

Angular Momentum

Angular momentum is the rotational analog to linear momentum.

The angular momentum of a particle, ℓ, is $\ell = mvr$. The direction of the angular momentum vector is perpendicular to the plane containing v and r.

Notice that just as Newton's second law may be written in terms of linear momentum:

$$\vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

it may also be written in terms of angular momentum:

$$\vec{\Gamma} = \frac{\Delta \vec{\ell}}{\Delta t}$$

Finally, angular momentum may be written in terms of moment of inertia and angular velocity for a rigid body and a fixed axis:

$$\vec{\ell} = I\vec{\omega}$$

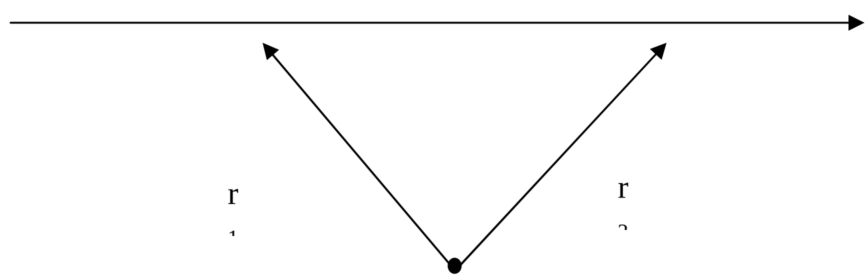
Newton's second law for translation/rotation:	$\vec{F} = m\vec{a}$	$\vec{\Gamma} = I\vec{\alpha}$
Work in translating/rotating systems:	$W = \vec{F} \cdot d\vec{s}$	$W = \Gamma d\theta$
Kinetic energy in translating/rotating systems:	$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
Power in translating/rotating systems:	$P = Fv$	$P = \Gamma\omega$
Momentum in translating/rotating systems:	$\vec{p} = m\vec{v}$	$\vec{\ell} = I\vec{\omega}$

Angular momentum is an enormously useful quantity in physics for several reasons. First, angular momentum is *conserved*, which means that in the absence of any external torques the angular momentum of a system remains constant.

$$\ell_i = \ell_f$$
$$mvr_i = mvr_f$$
$$I\omega_i = I\omega_f$$

Second, angular momentum may be computed in a wide variety of situations that, at first glance, don't involve rotational motion. All that it really necessary is to show motion with respect to a coordinate that one may compute angular momentum with respect to. In the case of instantaneous values this is normally an easy calculation. Complications may arise when computing, for instance, the angular momentum of a particle traveling in a straight line past a fixed coordinate over a short interval (over a long interval *r* changes with the displacement of the particle).

Notice that the instantaneous magnitude of angular momentum at each of the indicated



positions with respect to the origin is the same but that at any other point along the line the angular momentum with respect to the origin will have a different value. How would you calculate the value continuous value of the angular momentum for this system over a large interval?

Example 1. A man stands at the center of a turntable holding his arms extended horizontally with a 5 kg mass in each hand. He is set in motion with an angular velocity ω = 5 rev/s. Assume that the moment of inertia of a man is about 6 kg·m^2 and that his arms are 1 meter in length. What is his angular speed if he drops his arms to his sides resulting a final distance of the masses from his center of rotation of 0.2 meters?

$$I_{total} = I_{man} + I_{weights} = 6kg \cdot m^2 + mr^2$$

$$\rightarrow I_{initial} = 6kg \cdot m^2 + 10kg \cdot m^2 = 16kg \cdot m^2$$

$$\rightarrow I_{final} = 6kg \cdot m^2 + 0.4kg \cdot m^2 = 6.4kg \cdot m^2$$

Now conserve angular momentum:

$$I_i \omega_i = I_f \omega_f \rightarrow \frac{I_i \omega_i}{I_f} = \omega_f \rightarrow \frac{(16kg \cdot m^2)(5rev \cdot s^{-1})}{6.4kg \cdot m^2} = 12.5rev \cdot s^{-1}$$

Example 2. Our sun will eventually collapse from its current size into a much more compact white dwarf star, losing about half of it's mass in the a series of cataclysmic expansions and contractions that will precede this event. The radius of the white dwarf sun will be about 1% of its current value of 7×10^5 km. Although the sun has differential rotation a good average value for its rotational period is currently about 30 days. What will its rotational period be once the collapse into a white dwarf is complete? What does this imply about the properties of white dwarf stars in general?

$$\frac{1rev}{30days} = 0.033\, {}^{rev}\!/_{day}$$

Conserve angular momentum:

$$I_i\omega_i = I_f\omega_f \rightarrow \frac{I_i\omega_i}{I_f} = \omega_f$$

$$\frac{2}{5}MR^2\omega_i = \frac{2}{5}\frac{M}{2}\left(\frac{R}{100}\right)^2\omega_f$$

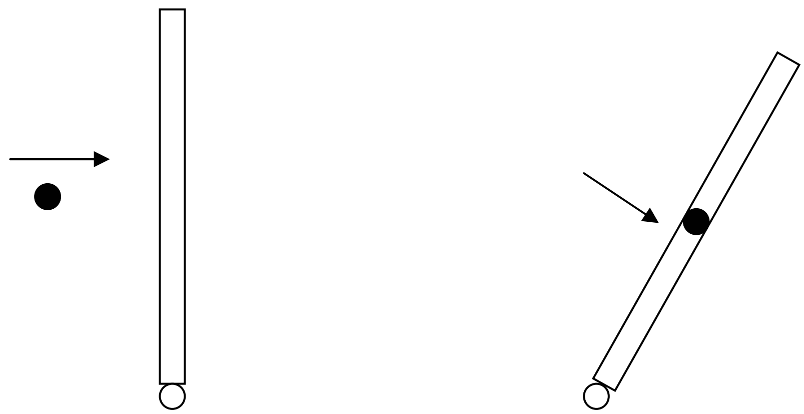
$$R^2\omega_i = \frac{1}{2}\left(\frac{R}{100}\right)^2\omega_f$$ (∵ 2/5 and *M* cancel)

$$\frac{2R^2\omega_i}{\left(\frac{R}{100}\right)^2} = \omega_f = \frac{(2)\left(7\times10^5\right)^2\left(0.033\, {}^{rev}\!/_{day}\right)}{\left(7\times10^3\right)^2} = 660\, {}^{rev}\!/_{day}$$

$$\frac{660rev}{day}\times\frac{1day}{1440\min} = 0.46\, {}^{rev}\!/_{\min} \approx 2.2\min$$

Example 3. A bullet, mass = 10 grams, is fired into the center of a door, mass = 15 kg, 1 meter wide with a velocity of 400 m/s. The door is mounted on frictionless hinges. Find the angular speed of the door after the impact.

Consider the type of collision involved here. There is no net external torque exerted on the bullet-door system so angular momentum *is* conserved. The bullet does exert a torque on the door but the door, in return exerts a torque on the bullet.



system before collision system after collision

Computing angular momentum with respect to the door hinge:

Before: $\ell_{bullet} = mvr = (0.01kg)(400m \cdot s^{-1})(0.5m) = 2.0kg \cdot m^2 \cdot s^{-1}$

$\ell_{door} = 0$

After: $\ell_{system} = I_{system}\omega$

$$I_{door} = \frac{ML^2}{3} = \frac{(15kg)(1.0m)^2}{3} = 5kg \cdot m^2$$

$$I_{bullet} = MR^2 = (0.01kg)(0.5m)^2 = 0.0025kg \cdot m^2$$

∵ conservation of angular momentum requires:

$$mvr = I_{system}\omega$$

$$2.0kg \cdot m^2 \cdot s^{-1} = (5.0025kg \cdot m^2)\omega$$

$$\omega = 0.4rad \cdot s^{-1}$$

Is energy conserved? $KE_i = \frac{1}{2}m_{sys}v^2 = 800J$ $KE_f = \frac{1}{2}I\omega^2 = 0.4J$ - **1/2000 of the initial value!**

Gyroscopes and Tops

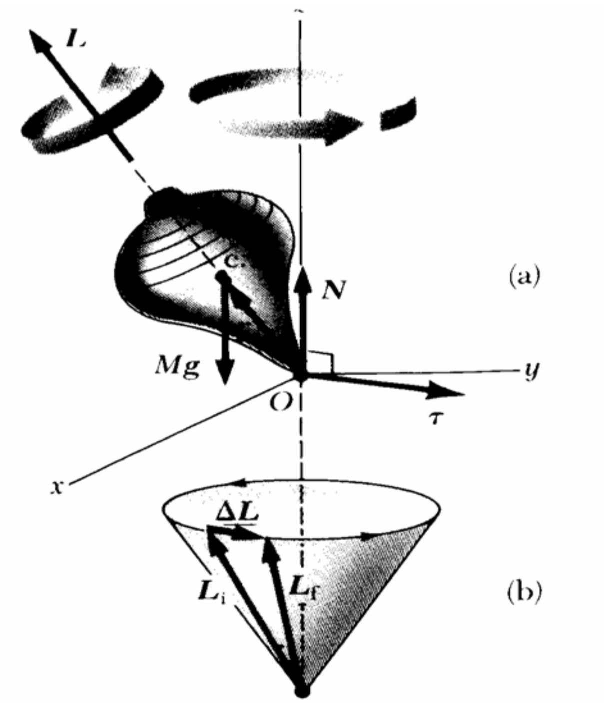


Figure 11.19 Precessional motion of a top spinning about its axis of symmetry. The only external forces acting on the top are the normal force, ***N***, and the force of gravity, ***Mg***. The direction of the angular momentum, ***L***, is along the axis of symmetry.

From *Physics for Scientists and Engineers*, Serway, 3rd Ed.

Consider the top illustrated above. When a top is first "spun up" if it is done so with sufficient angular velocity and oriented so that it is upright it stays in that position as it spins. By virtue of its spin the top produces an angular momentum vector, *L*, that points along the axis of spin. For a while the top, because of its rotational inertia, is stable and is not pulled over by the force of gravity (a condition known as *sleeping*). At some point, however, it slows below a certain angular velocity known as the *critical speed* and begins to topple. Before the top topples completely over it precesses, or begins a canted orbit around the original spin axis of the system as shown above.

Since $\tau = \frac{\Delta \vec{L}}{\Delta t}$, the torque produced by gravity which becomes non-zero as soon as the top begins to tip over produces a change in the angular momentum of the system which is in the same direction as that of τ, i.e., perpendicular to the plane containing *r*, *Mg*, and *L*.

Note that the magnitude of L does not change (at least not very rapidly), rather it is the direction of *L* that changes resulting in "wobble" or precessional motion, or the deflection of the angular momentum vector, *L*, about the original spin axis of the system.

Angular Momentum as a Fundamental Unit

Angular momentum plays a role in microscopic as well as macroscopic physics. In quantum mechanics the fundamental unit of angular momentum is related to another unit known as Planck's constant: $h = 6.626\times 10^{-34}\, J\cdot s$, The fundamental unit of angular momentum $\hbar = 1.054\times 10^{-34}\, \frac{kg\cdot m^2}{s^2}$, which is $h/2\pi$. Of great importance is the notion that angular momentum is *quantized*, i.e., that all values of angular momentum occur in integer multiples of this fundamental value even for macroscopic systems. Note that this fundamental unit has an exceedingly small value. What does this imply about quantization of angular momentum for macroscopic systems?