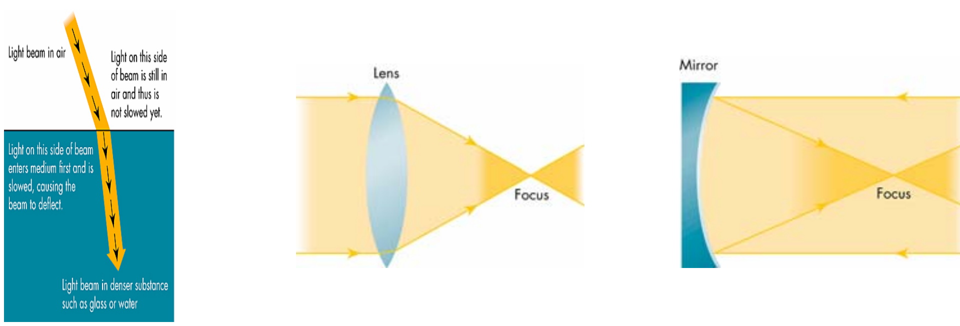
Geometric Optics – Objects, Images, Rays

The progress of rays will be used to determine image formation for lenses, mirrors and other optical elements.

- An *object* is anything that is being viewed, e.g., when one looks at a tree through a lens, with a mirror or any other optical device the tree is referred to an optical object.
- *Object Distance*, *s*, is the distance from an object to an optical element.
- An *image* is the likeness of an object produced at a point in space by a lens, mirror or other optical device.
- *Image Distance*, *s'*, is the distance from a lens or mirror to an image.
- *Magnification, m,* is the amount of size increase (or decrease) of the image as compared to the object. $m = h_i / h_o$
- Images are formed by lenses and mirrors where light rays cross or focus.
- A *lens* is an optical device that uses *refraction* through curved surfaces to focus light.
- A *mirror* is an optical device that uses reflection to focus light.
- A *converging lens* is a lens with two convex surfaces.
- A *diverging lens* is a lens with two concave surfaces.

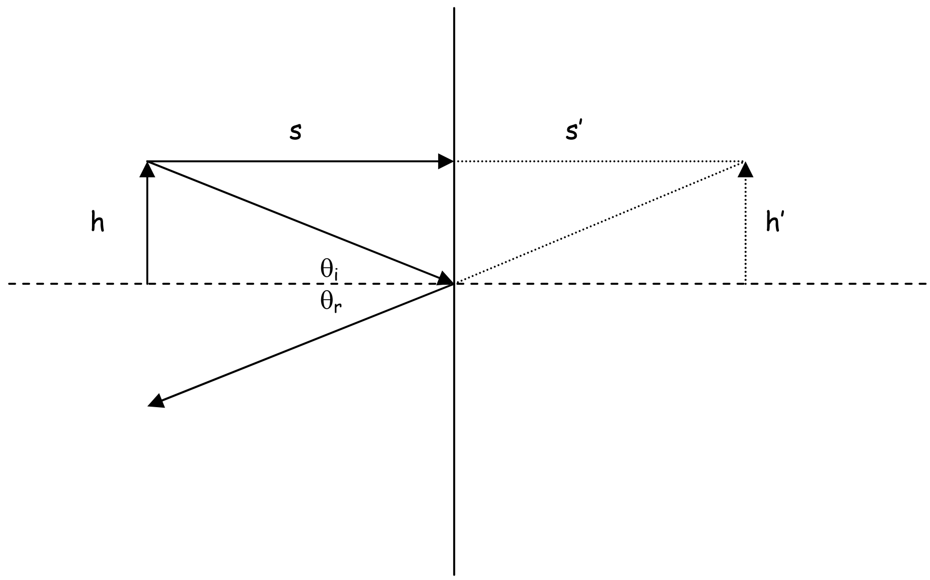


Courtesy of Thomas Arny

- A real object is one in which light rays physically emanate from the object.
- A real image is one in which light rays physically intersect at the image location.
- A virtual object is one from which light rays appear to emanate but physically do not.
- A virtual image is one in which light rays do not physically intersect at the image point but appear to diverge from that point.
- Real images may be displayed on a screen while virtual images may not.
- All optical elements and optical systems are *reversible*.
- The objective of any optical system is to cause rays of light to cross and produce an image.

Images Formed by Plane Mirrors

A highly reflective plane mirror is the simplest reflecting surface.



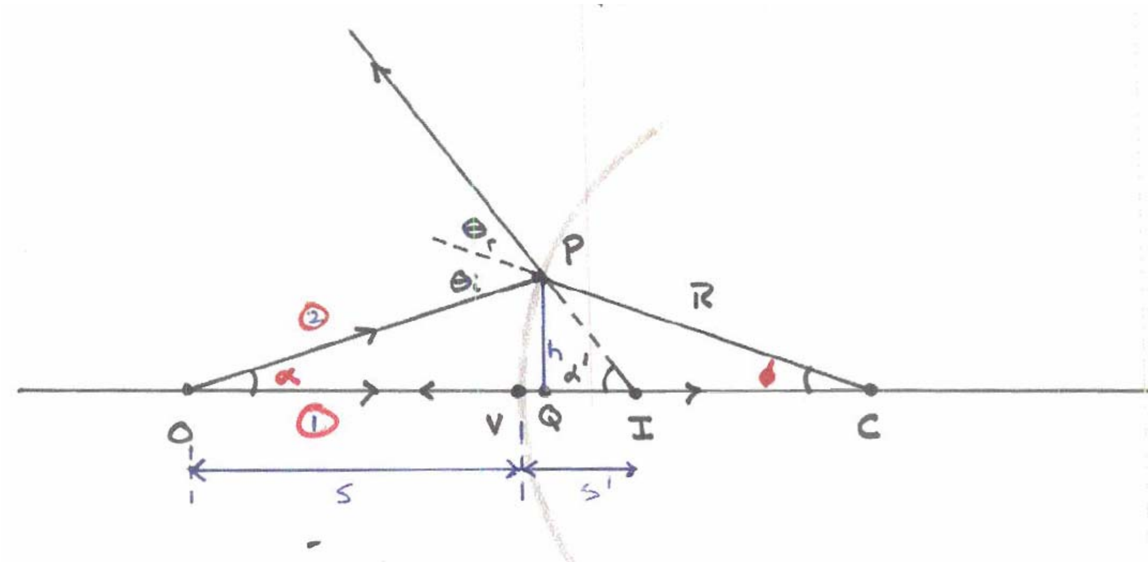
- Plane mirrors produce virtual images for real objects.
- Plane mirrors show left-right reversal
- $s = s'$
- $h = h'$
- $m = 1$

Images formed by Spherical Mirrors

We'll consider two types of spherical mirrors: concave and convex.

- Convex mirrors curve outward toward the object
- Concave mirrors curve inward

Consider a convex mirror of constant radius (spherical).



- Ray **1** proceeds from the object at ***O*** to the vertex of the mirror at **V** along the optic axis and is reflected back along the optic axis.
- Ray 2 emanates at an arbitrary angle from ***O*** and is incident on the mirror at point **P**.
- **I** is a virtual image located by virtual rays **1** and **2** (note that the rays diverge as they leave the surface of the mirror)

We seek a relationship between s and s' that relates to the only adjustable parameter in the system, **R**, the radius of curvature of the mirror.

To find this relationship we'll compare the interior angles and associated exterior angles in the diagram above.

- We'll employ the *paraxial approximation* commonly used in geometrical optics which means that we'll assume that all rays approximately parallel to the optical axis of the system.
- The paraxial approximation is a simplifying set of assumptions that greatly simplifies the derivation of the needed relationship without affecting the accuracy of the result.

① $\Delta OIP \quad \alpha + \alpha' = 2\theta$

② $\Delta OCP \quad \alpha + \phi = \theta$

The exterior angle of a triangle equals the sum of its interior angles

② → ① $\quad 2(\alpha + \phi) = \alpha + \alpha'$

$$2\alpha + 2\phi = \alpha + \alpha'$$

③ $\quad \alpha - \alpha' = -2\phi$

Paraxial Approximation: (Small angles) $\sin\alpha \approx \tan\alpha \approx \alpha$

Could use sin or tan !

$\Delta OQP \quad \tan\alpha = \dfrac{h}{\overline{OQ}}$ $\quad$ $\tan\alpha \approx \alpha$ in radians

$\Delta IQP \quad \tan\alpha' = \dfrac{h}{\overline{IQ}}$ $\quad$ $\tan\alpha' \approx \alpha'$ in radians

$\Delta CQP \quad \tan\phi = \dfrac{h}{\overline{CQ}}$ $\quad$ $\tan\phi \approx \phi$ in radians

$$\therefore \quad \alpha \approx \frac{h}{OQ} \approx \frac{h}{s}$$

$$\alpha' \approx \frac{h}{IQ} \approx \frac{h}{s'}$$

$$\phi \approx \frac{h}{CQ} \approx \frac{h}{R}$$

for small angles

$$\Rightarrow \quad \frac{h}{s} - \frac{h}{s'} = -2\frac{h}{R}$$

Cancellation of h throughout yields:

$$\frac{1}{s} - \frac{1}{s'} = -\frac{2}{R} \quad \text{(equation 1)}$$

Notice that if the reflecting surface had been concave instead of convex the terms in equation 1 would have been positive. It is possible to adapt a sign convention by which both cases may be represented:

$$\frac{1}{s} + \frac{1}{s'} = -\frac{2}{R} \quad \text{(equation 2)}$$

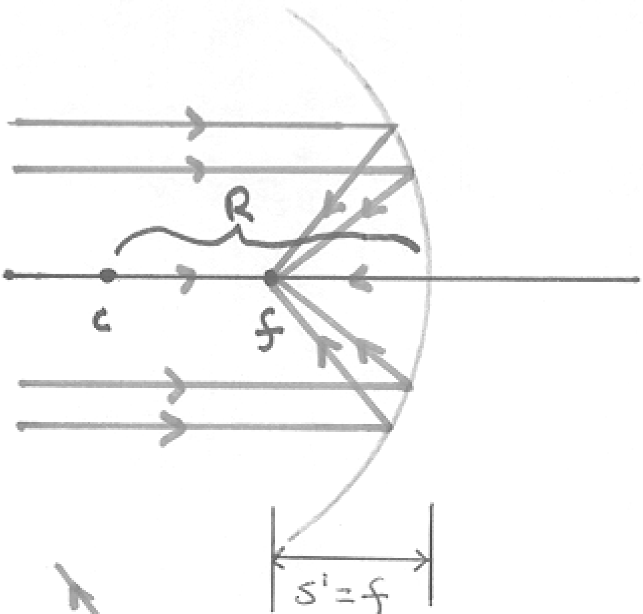
This sign convention assumes that if light propagates from left to right:

1. "s" is positive when O is left of V (real object). "s" is negative when O is right of V (virtual object).

2. "s'" is positive when I is left of V (real image). "s'" is negative when I is right of V (virtual image).

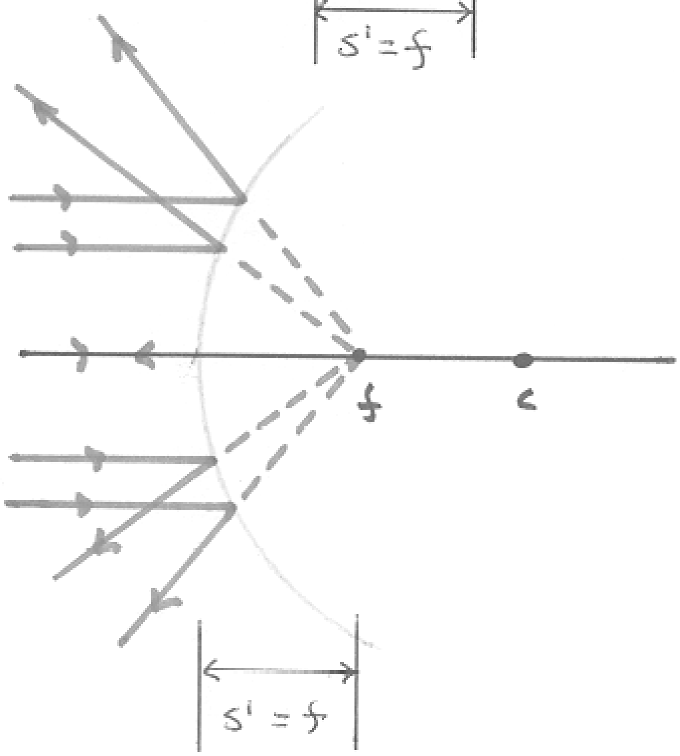
3. "R" is positive when C is right of V (convex), negative when C is left of V (concave).

- Positive object and image distances correspond to real objects and real images and convex mirrors have positive radii of curvature.
- Applying #2 to equation 1 results in equation 1 becoming identical to equation 2.
- Virtual objects occur only in systems of two or more elements
- The spherical equation described by equation 2 yields for $R = \infty$ the same result found previously for a plane mirror, $s = s'$.
- Notice also that s and s' are interchangeable in equation 2 as conjugate points.
- If $s = \infty$ then the rays from the object are parallel and $s' = -\frac{R}{2}$



$$\frac{1}{\infty} + \frac{1}{f} = -\frac{2}{R}$$

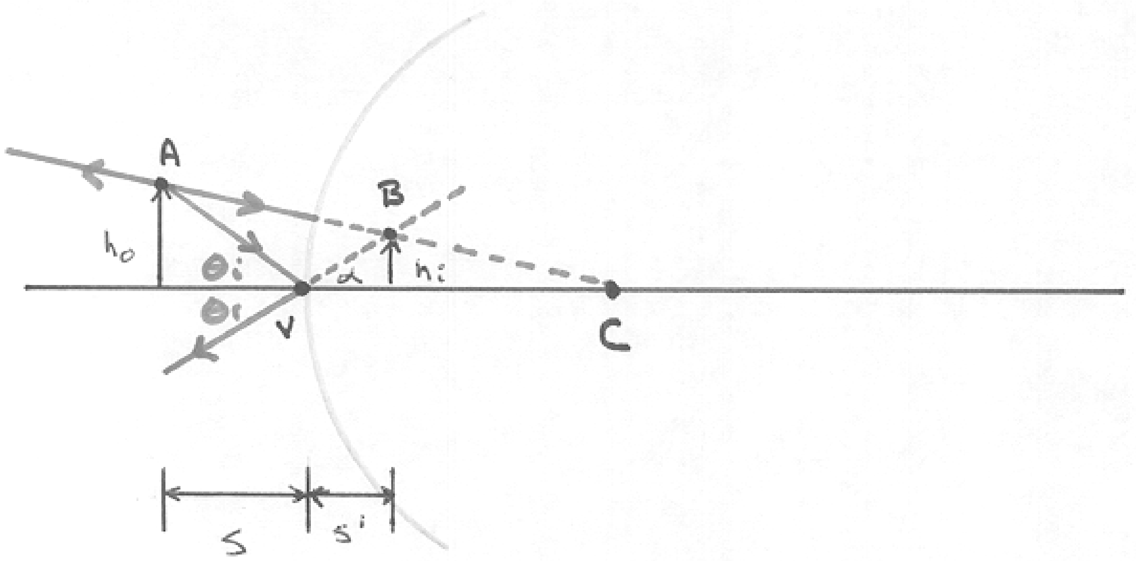
$$\frac{1}{f} = -\frac{2}{R}$$



f is defined as the focal length of the mirror $f = -\frac{R}{2}\begin{cases} > 0, & concave \\ < 0, & convex \end{cases}$

The mirror equation may be written: $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$

To determine magnification of the image we consider the following:



$$\tan\theta_i = \frac{h_o}{AV} \approx \frac{h_o}{s} \qquad \text{(small } \angle \text{ approx.)}$$

$$\tan\alpha = \frac{h_i}{BV} \approx \frac{h_i}{s'} \qquad \text{(small } \angle \text{ approx.)}$$

$$\therefore \frac{h_o}{s} = \frac{h_i}{s'}$$

Define: $$m = \frac{h_i}{h_o} = \frac{s'}{s}$$

Where:

- m (+) when image has same orientation as object
- m (-) when image has opposite orientation as object

Notice that in order to provide a + magnification in the figure above where s' is negative we modify the magnification equation above to:

$$m = \frac{-s'}{s} \qquad \text{(the magnification equation)}$$

Images Formed by Single Surface Plane Refraction

A short topic – there aren't any! Why? Plane refraction can displace rays of light from their path, but all beams are displaced by the same amount and at the same angle of refraction. Beams of light, therefore, will not cross due to plane refraction.

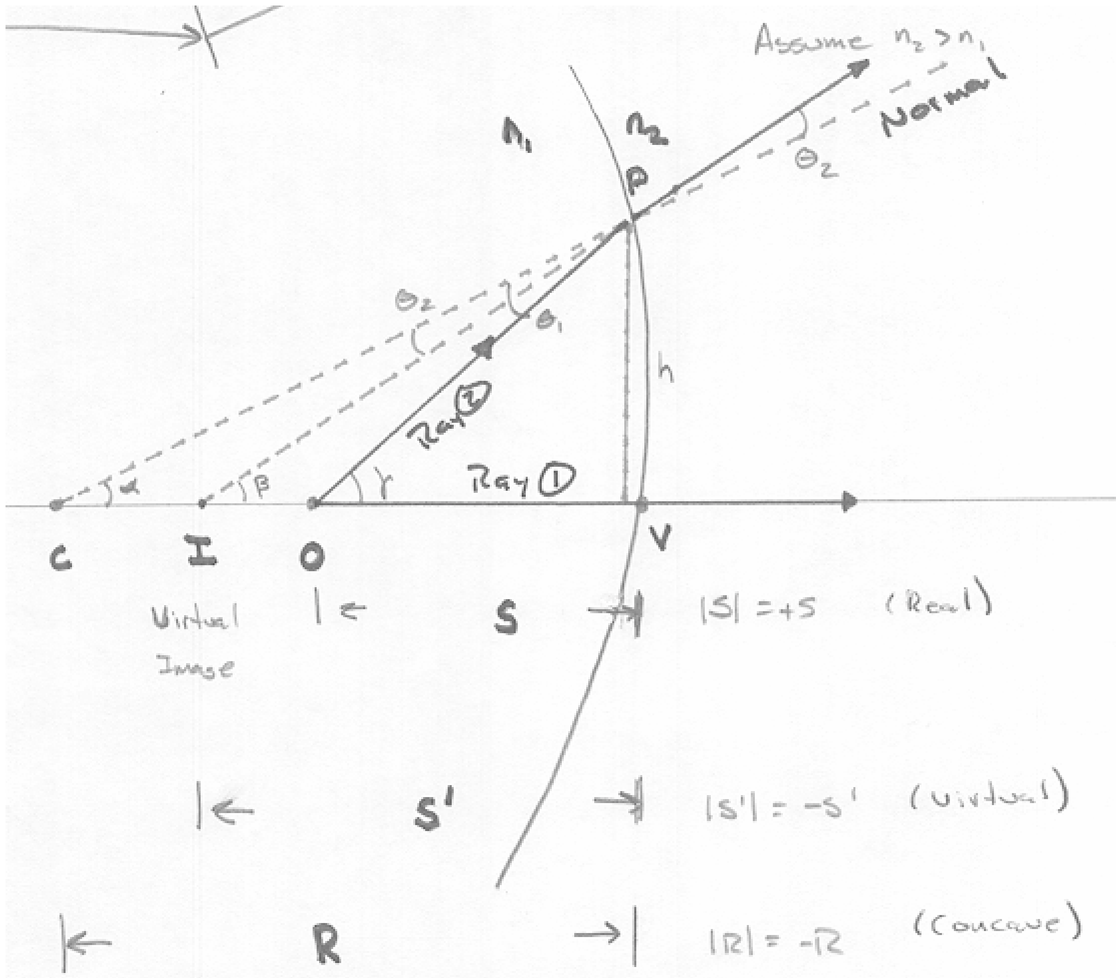
- Plane refraction is still important in multi-surface optical systems
- Mirages involve plane refraction.

Images Formed by Single Surface Spherical Refraction

The focal length, *f*, of a refracting surface is a defining characteristic of that surface.

- The focal length of a refracting surface may be defined as the distance from the surface at which a distant object will produce an image.
- As with mirrors, the focal length of a refracting surface depends on the radius of curvature of the spherical surface.
- We seek a relationship between the object and image distances, the radius of the refracting surface, and the indicies of refraction.

$$\theta_1 = \theta_i$$
$$\theta_2 = \theta_t$$



Looking at the respective indicies of refraction it is clear that;

- If $n_1 > n_2$ $\quad \sin\theta_2 > \sin\theta_1 \Rightarrow \theta_2 > \theta_1$
- If $n_1 < n_2$ $\quad \sin\theta_2 < \sin\theta_1 \Rightarrow \theta_2 < \theta_1$

$\Delta CIP \qquad \beta = \alpha + \theta_2 \Rightarrow \theta_2 = \beta - \alpha \qquad$ (1)

$\Delta COP \qquad \gamma = \alpha + \theta_1 \Rightarrow \theta_1 = \gamma - \alpha \qquad$ (2)

$\tan\alpha \cong \frac{h}{-R} \cong \alpha \qquad$ (paraxial) (3)

$\tan\beta \cong \frac{h}{-s'} \cong \beta \qquad$ (paraxial) (4)

$\tan\gamma \cong \frac{h}{s} \cong \gamma \qquad$ (paraxial) (5)

$n_1\theta_1 = n_2\theta_2 \qquad$ (Snell's paraxial) (6)

(3) + (4) $\rightarrow$ (1) $\qquad \theta_2 = -\frac{h}{s'} + \frac{h}{R}$

(3) + (5) $\rightarrow$ (2) $\qquad \theta_1 = \frac{h}{s} + \frac{h}{R}$

$\rightarrow$ (6) $\rightarrow \qquad n_1 h\left(\frac{1}{s} + \frac{1}{R}\right) = n_2 h\left(\frac{1}{R} - \frac{1}{s'}\right)$

$$n_1\left(\frac{1}{s} + \frac{1}{R}\right) = n_2\left(\frac{1}{R} - \frac{1}{s'}\right)$$

$$\frac{n_1}{s} + \frac{n_1}{R} = \frac{n_2}{R} - \frac{n_2}{s'}$$

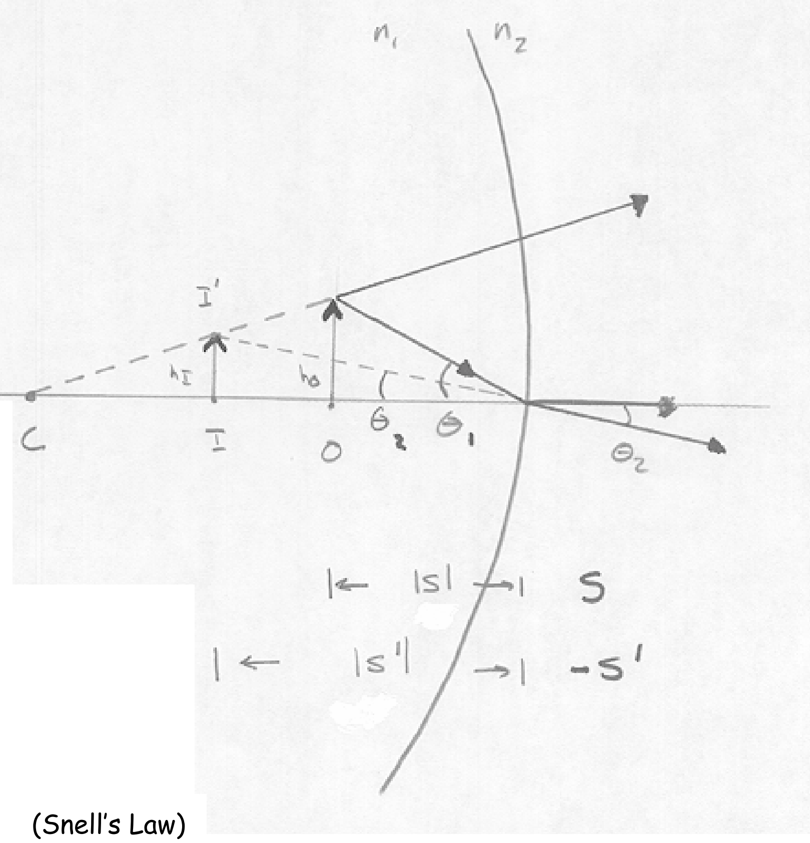
$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$$ (single surface spherical refraction equation)

The latter is the equation we were looking for.

Notice that for a plane surface, $R = \infty$, and: $\frac{n_1}{s} = -\frac{n_2}{s'} \rightarrow s' = -\left(\frac{n_2}{n_1}\right)s$

- Real objects form virtual images for plane refracting surfaces when the index of refraction differs on each side of the refracting surface

To determine the magnification, m, for a refracting surface consider the following:



$$\sin\theta_2 = \frac{h_i}{s'}$$

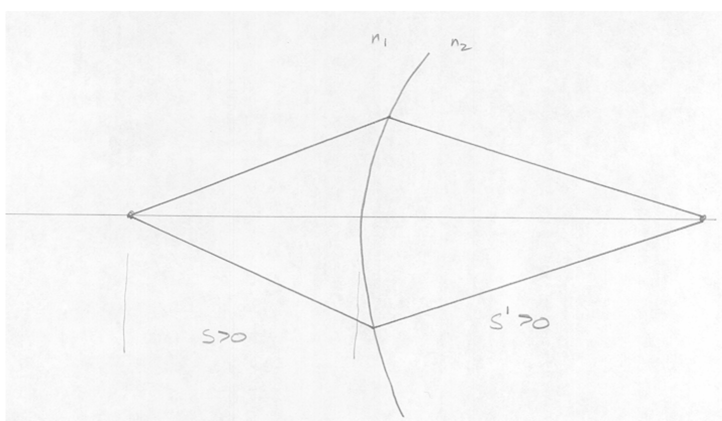
$$\sin\theta_1 = \frac{h_o}{s}$$

$$m = \frac{h_i}{h_o} = -\left(\frac{\sin\theta_2}{\sin\theta_1}\right)\frac{s'}{s}$$

$$\frac{\sin\theta_2}{\sin\theta_1} = -\frac{h_i}{h_o}\frac{s}{s'} = \frac{n_1}{n_2}$$ (Snell's Law)

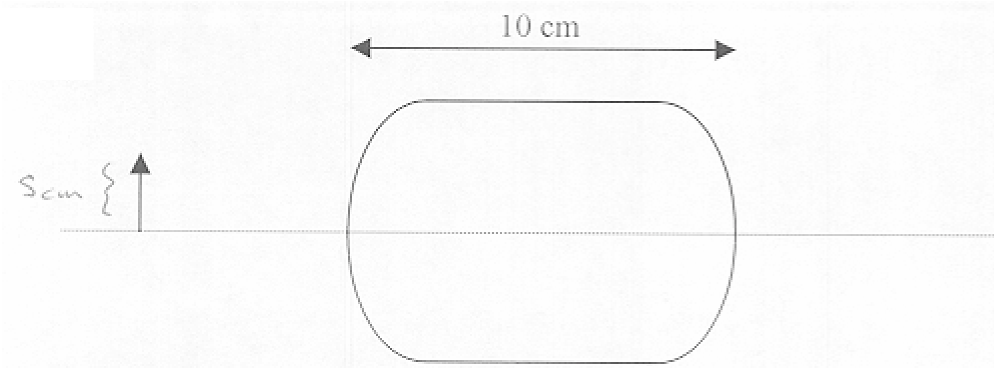
$$m = \frac{h_i}{h_o} = -\frac{n_1}{n_2}\frac{s'}{s}$$ (magnification for a single refracting surface)

- The figure on the right is a ray trace for a convex refracting surface. Note the conditions for real and virtual objects/images.



- Multiple refracting surfaces require repeated application of the relationships derived above with the output from one step used as the input to the next.

Example 1 Consider the following lens where $R_1 = R_2 = |5|cm$. The object is 5 cm tall and located 30cm left of the first refracting surface. The index of refraction of the material in the lens is 4/3.



Step 1 – Refraction

At the first surface $R = +5cm, s = +30cm, n_1 = 1, n_2 = 4/3$

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \rightarrow \frac{1.0}{30cm} + \frac{4}{3s'} = \frac{\frac{4}{3} - 1.0}{5cm} \rightarrow s' = 40cm$$

$$m = \frac{h_i}{h_o} = -\frac{n_1}{n_2}\frac{s'}{s} \rightarrow -\left(\frac{3}{4}\right)\left(\frac{40cm}{30cm}\right) = -1$$

The first surface is trying to form a real, inverted, 5 cm tall image 40 cm right of the surface but the light is intercepted by the second surface. We proceed as before using the image from the first surface as the object for the second surface.

At the second surface: $R = -5cm, s = -30cm, n_1 = 4/3, n_2 = 1$ Notice that the object is virtual.

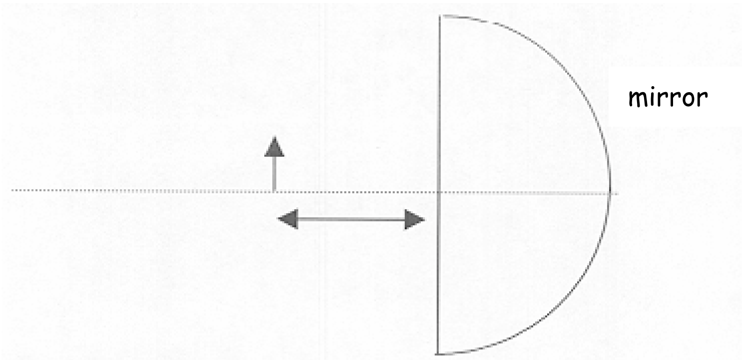
$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \rightarrow \frac{\frac{4}{3}}{-30cm} + \frac{1}{s'} = \frac{1.0 - \frac{4}{3}}{-5cm} \rightarrow s' = 9cm$$

$$m = \frac{h_i}{h_o} = -\frac{n_1}{n_2}\frac{s'}{s} \rightarrow -\left(\frac{4}{3}\right)\left(\frac{9cm}{-30cm}\right) = +0.4$$

$$M = m_1 m_2 \rightarrow (-1)(0.4) = -0.4$$

The final image is real, located 9 cm right of the second refracting surface, is inverted, and 2 cm tall.

Example 2 An object, 5 cm tall, is located in air 10 cm left of the hemispherical lens shown below. The medium inside of the hemispherical lens has an index of refraction of 1.5. The back surface of the hemispherical lens is mirrored and has a constant radius of 10 cm. Describe the final image.



Step 1 - First Refraction at plane surface

$$R = \infty, \quad s = +10cm, \quad h_o = +5cm, \quad n_1 = 1.0, \quad n_2 = 1.5$$

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \rightarrow \frac{1.0}{10cm} + \frac{1.5}{s'} = \frac{1.5 - 1.0}{\infty} \rightarrow s' = -15cm$$

$$m = \frac{h_i}{h_o} = -\frac{n_1}{n_2}\frac{s'}{s} \rightarrow -\left(\frac{1.0}{1.5}\right)\left(\frac{-15cm}{10cm}\right) = 1$$

The image is located 15 cm *left* of the refracting surface, is virtual, has the same orientation in space as the object (upright) and is the same size as the object.

Step 2 - Reflection at curved surface

Notice that the *virtual image* from the first step is a *real object* for the second and is located 25 cm left of the mirror (*s* = +25 cm).

$$\frac{1}{s} + \frac{1}{s'} = -\frac{2}{R} \rightarrow \frac{1}{25} + \frac{1}{s'} = -\frac{2}{-10} \rightarrow s' = +6.25cm$$

$$m = -\frac{s'}{s} = -\frac{6.25cm}{25cm} = -0.25$$

The image is real and located 6.25 cm *left* of the mirror (the reflection reverses the direction of flow in the system so a real image must be on the same side of the mirror as the incident light), is inverted with respect to the object and 25% the size of the object

Step 3 - Second refraction at plane surface

The real image from step 2 is a real object for step 3.

$$R = \infty, \quad s = +3.75cm, \quad n_1 = 1.5, \quad n_2 = 1.0$$

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \rightarrow \frac{1.5}{3.75cm} + \frac{1.0}{s'} = \frac{1.0 - 1.5}{\infty} \rightarrow s' = -2.5cm$$

$$m = \frac{h_i}{h_o} = -\frac{n_1}{n_2}\frac{s'}{s} \rightarrow -\left(\frac{1.5}{1.0}\right)\left(\frac{-2.5cm}{3.75cm}\right) = 1$$

The image is virtual, located 2.5 cm right of the plane surface, is the same size and orientation in space as the object.

The total magnification of the system is the product of the magnification from each step:

$$M = m_1 m_2 m_3 \rightarrow (1)(-0.25)(1) = -0.25$$

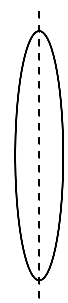
The final image is located 2.5 cm right of the plane surface, is inverted, virtual and 1.25 cm tall.

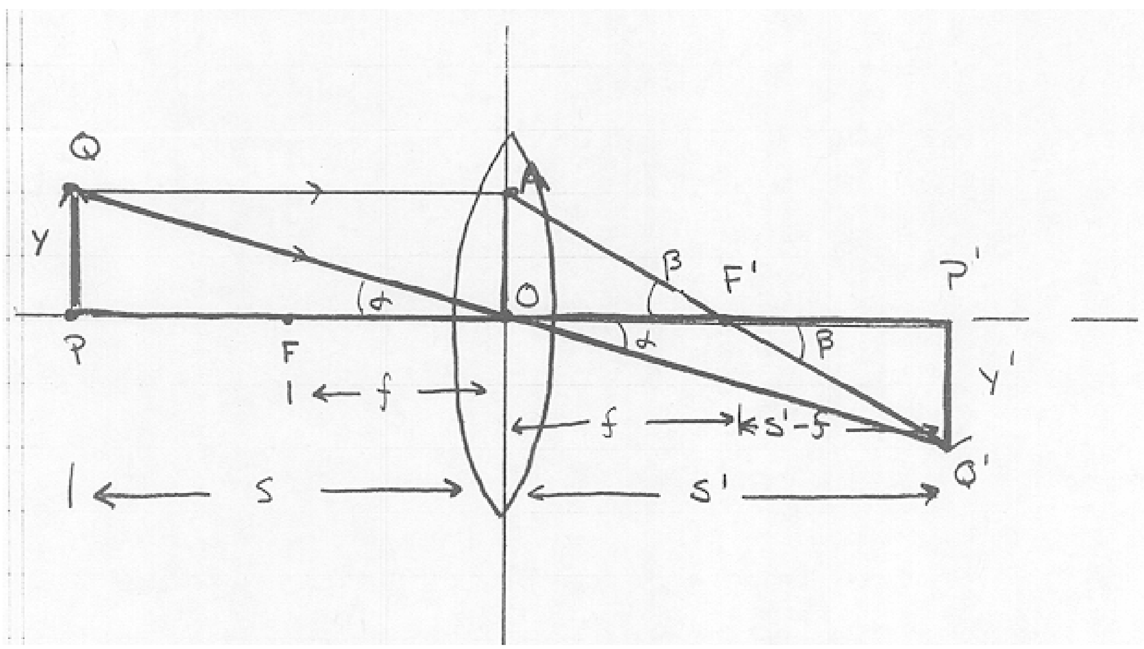
Thin Lenses

The thin lens approximation allows us to treat most lenses as a single refracting surface. In order to apply the thin lens approximation:

- The lens must be thin
- The lens must be surrounded by a common index of refraction
- The focal length of the lens must be known (or must be determined)

When the above conditions are met both refracting surfaces of a lens may be treated as a single refracting surface located at the plane that passes through the center of the lens:





We seek a relationship between *s*, *s'*, and *f*.

Notice that triangles PQO and P′Q′O are similar: $\rightarrow \frac{y}{s} = -\frac{y'}{s'}$ and $\frac{y'}{y} = -\frac{s'}{s}$ (1)

Triangles OAF′ and P′Q′F′ are similar: $\rightarrow \frac{y}{f}=\frac{-y'}{s'-f}$ and $\frac{y'}{y}=-\frac{s'-f}{f}$ (2)

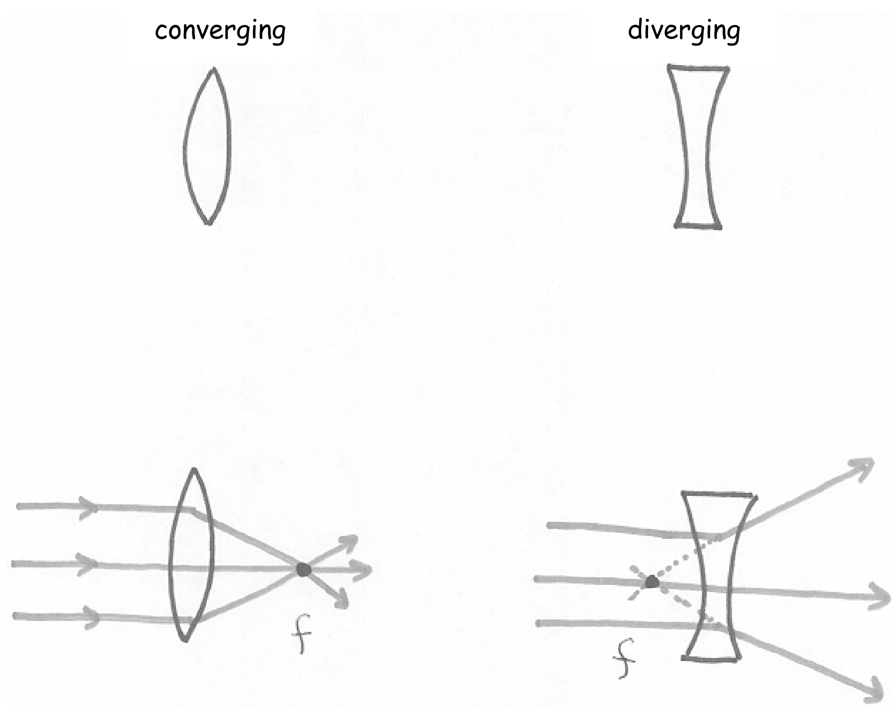
Equating (1) and (2): $$\frac{s'}{s}=\frac{s'-f}{f}$$

Now dividing through by s': $$\frac{1}{s}=\frac{1-\frac{f}{s'}}{f}\rightarrow\frac{1}{s}=\frac{1}{f}-\frac{1}{s'}$$

This yields the *thin lens equation*: $$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

It may be shown that magnification is given by: $m=\frac{-s'}{s}$

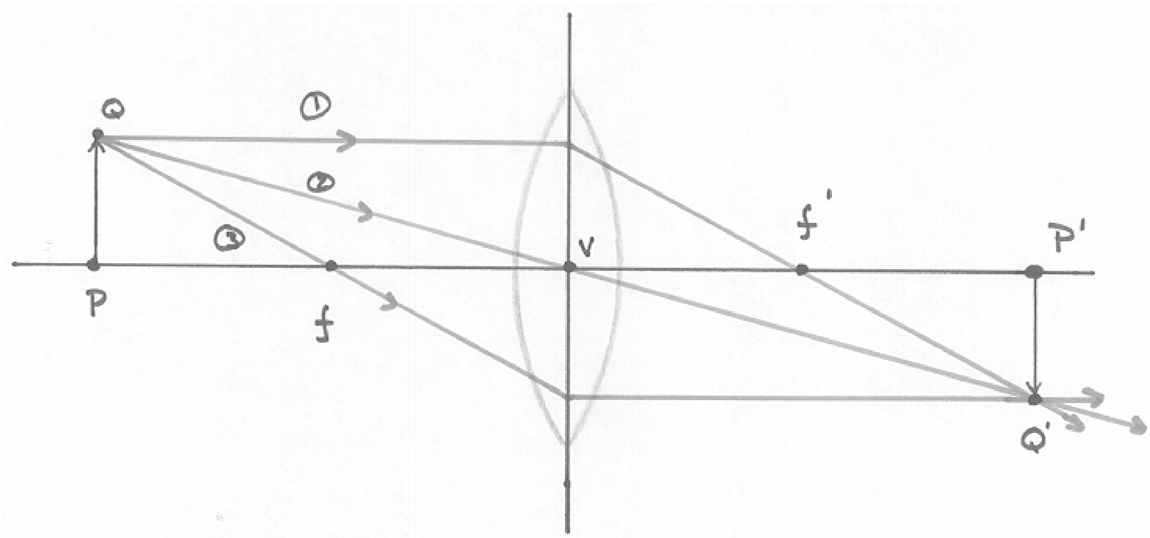
- The thin lens equations are the same as the mirror equations
- The thin lens equations apply to all thin lenses converging or diverging or even asymmetric as long as the focal length is known.



Notice that when $s=\infty$, $\frac{1}{\infty}+\frac{1}{s'}=\frac{1}{f}\rightarrow s'=f$, so the focal length and image distance coincide for faraway objects.

Ray Tracing with Principal Rays

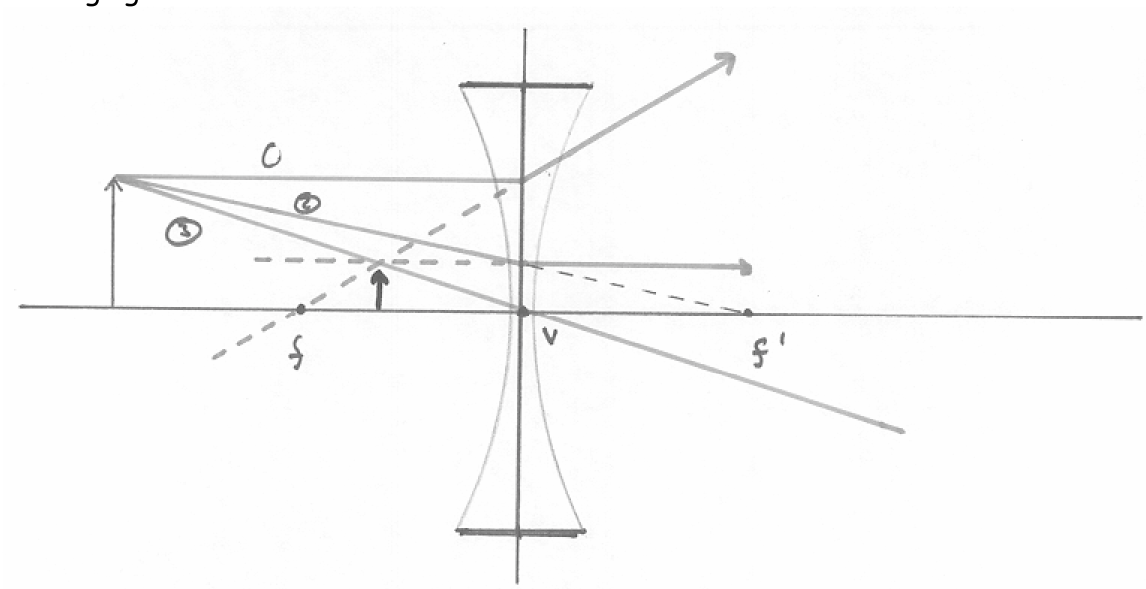
The converging lens



	Toward Lens	From Lens
Principal Rays:	(1) parallel to optic axis	through *f*
	(2) through the Vertex	undeviated
	(3) through *f*	parallel to the optic axis

Any two of these rays will locate the image and define its size.

The diverging lens



	Toward Lens	From Lens
Principal rays:	(1) parallel to optic axis	virtual ray through *f*
	(2) to *f'*	virtual ray parallel to the optic axis
	(3)to *v*	undeviated

Example 3 An object 4 cm tall is located 1 meter left of a lens of f = +10 cm. Describe the image produced.

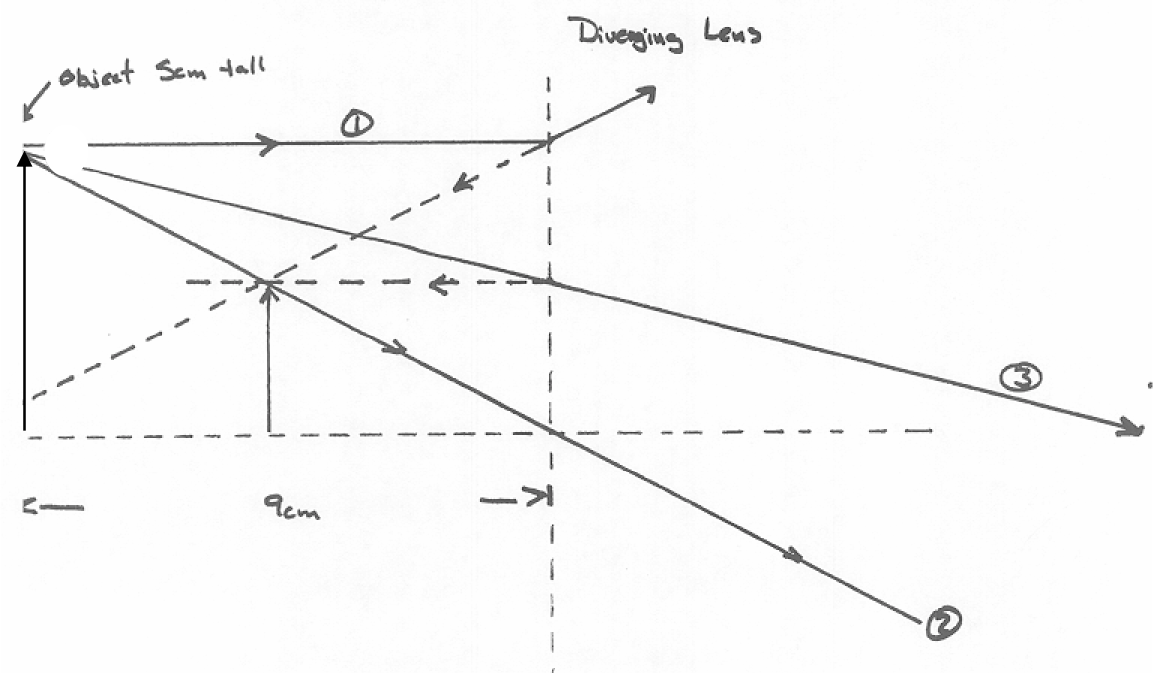
$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}\rightarrow\frac{1}{100cm}+\frac{1}{s'}=\frac{1}{10cm}\rightarrow s'=11.1cm$$

$$m=-\frac{s'}{s}=-\frac{11.1}{100cm}=-0.11$$

The image is real, located 11.1 cm right of the lens, is inverted and 11% the height of the object or 0.44cm.

Example 4 An object 5 cm tall is located 9 cm left of a lens of focal length –10cm. Use graph paper and a ray trace to locate the final image. Describe the final image.

One may use the thin lens equations to verify that the final image is virtual, erect, 2.5 cm tall, and is located 4.7 cm to the left of the lens.



Example 5 Show that for a converging lens, unit magnification (*m* = -1) results when *s* = 2f.

From our thin lens derivation $m = \frac{-s'}{s} = -\frac{1}{s}\left(\frac{1}{\frac{1}{f} - \frac{1}{s}}\right)$, if $s = 2f$ then $m = \frac{1}{2f}\left(\frac{1}{\frac{1}{f} - \frac{1}{2f}}\right)$

$$m = -\frac{1}{2f}\left(\frac{1}{\frac{1}{2f}}\right) = -\frac{1}{2f}2f = -1$$

Under what condition is *m* = +1 approached?

If $m = +1$ then $-\frac{1}{s} = \frac{1}{f} - \frac{1}{s'}$

This is true when $f \rightarrow \infty$

The Lens Maker's Equation

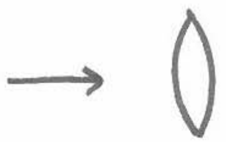
The lens maker's equation works for any lens that cannot be treated with the thin lens equation.

Recall the general equation for refraction at a spherical surface: $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

It may be shown that: $\frac{1}{s} + \frac{1}{s'} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ or $\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

If the lens is surrounded by air: $\frac{1}{f} = (n_2 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

Example 6 Consider a lens of unknown focal length in air. If the lens is symmetric with radii of 10 cm compute the focal length.



$$|r_1| = |r_2| = 10\text{ cm}$$

$$n = 1.52$$

Find f

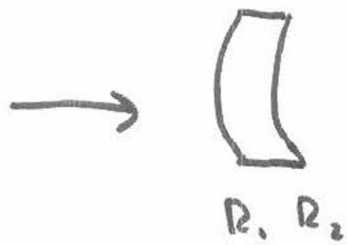
$$\frac{1}{f} = 1.52 - 1 \left(\frac{1}{10\text{cm}} - \frac{1}{-10\text{cm}} \right)$$

$$f = 9.62\text{cm}$$

Notice that working the problem from right to left has no outcome on the answer. This is due to the principle of reversibility.

The focal length of a lens is the same no matter which side light approaches from. This applies to *all* lenses.

Example 7 Consider an asymmetric lens of unknown focal length in air. If the lens has surfaces with radii or 10 cm and 15 cm as shown below, compute the focal length.



$R_1 = 10\text{ cm}$

$R_2 = 15\text{ cm}$

$n = 1.52$

$$\frac{1}{f} = (1.52-1)\left(\frac{1}{10\text{cm}} - \frac{1}{15\text{cm}}\right)$$

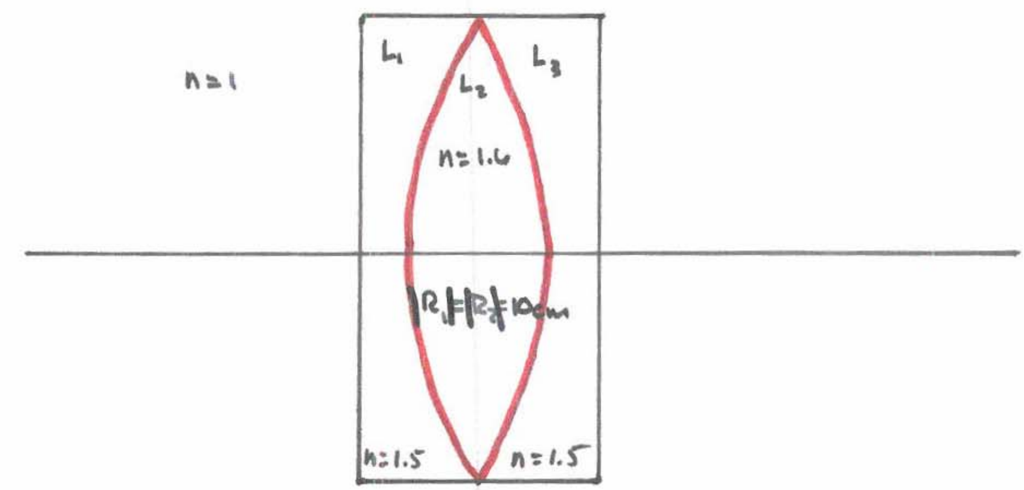
$$f = 57.7\text{ cm}$$

Backwards

$$\frac{1}{f} = (1.52-1)\left(\frac{1}{-15\text{cm}} - \frac{1}{-10\text{cm}}\right)$$

$$f = 57.7\text{ cm}$$

Example 8 Consider an oil lens between two glass lenses in air as shown below. Determine the focal length of the combination.



This is an example of a compound lens system, i.e., one in which lenses are in direct physical contact with each other. Since the system is surrounded by air we'll compute the focal length of each lens separately in air and then compute the focal length of the combination. Note that $|R_1| = |R_2| = 10cm$.

① $$\frac{1}{f} = (n_2 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{\infty} - \frac{1}{10cm}\right)$$

$$f = -20\ cm$$

(2) 

$$\frac{1}{f} = (1.6-1)\left(\frac{1}{10\,cm} - \frac{1}{-10\,cm}\right)$$

$$f = 8.33\ cm$$

(3)  by reversibility, $f = -20\ cm$

For any compound lens it may easily be shown that the focal length of the combination is the sum of the focal lengths of the individual lenses as shown below:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

$$= -\frac{1}{20} + \frac{1}{8.33} - \frac{1}{20}$$

$$f = +50\ cm$$

Summary of Reflective, Refractive and Lens Equations

- The mirror equation relates object distance (*s*), image distance (*s'*), radius of curvature (*R*), magnification (*m*) and focal length (*f*) as follows:

$$\frac{1}{s}+\frac{1}{s'}=-\frac{2}{R}=\frac{1}{f}\ ,\ m=-\frac{s'}{s}$$

- For spherical refractive surfaces the object distance (*s*), image distance (*s'*), radius of curvature (*R*), magnification (*m*) and indicies of refraction (*n*) are related as follows:

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{n_2-n_1}{R}\ ,\ m=-\frac{n_1}{n_2}\frac{s'}{s}$$

- The thin lens equation relates object distance (*s*), image distance (*s'*), magnification (*m*) and focal length (*f*) as follows:

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}\ ,\ m=-\frac{s'}{s}$$

- The Lens Maker's equation for any lens that cannot be treated like a thin lens is:

$$\frac{1}{f}=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\ \text{or}\ \frac{1}{f}=(n_2-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

Other Optical System Errata

- The *diameter* of a lens or mirror determines its light gathering power
- The ratio of focal length to diameter determines the brightness of the image. This ratio is called the *f-number* of the lens:

$$f_{number}=\frac{f}{diameter}$$

  A large *f-number* means that the image is *not* very bright.
- The *principle of reversibility* requires that the flow of light rays through any optical system be the same in the forward or reverse directions.
- Compound lenses are lenses that are in direct physical contact with each other.