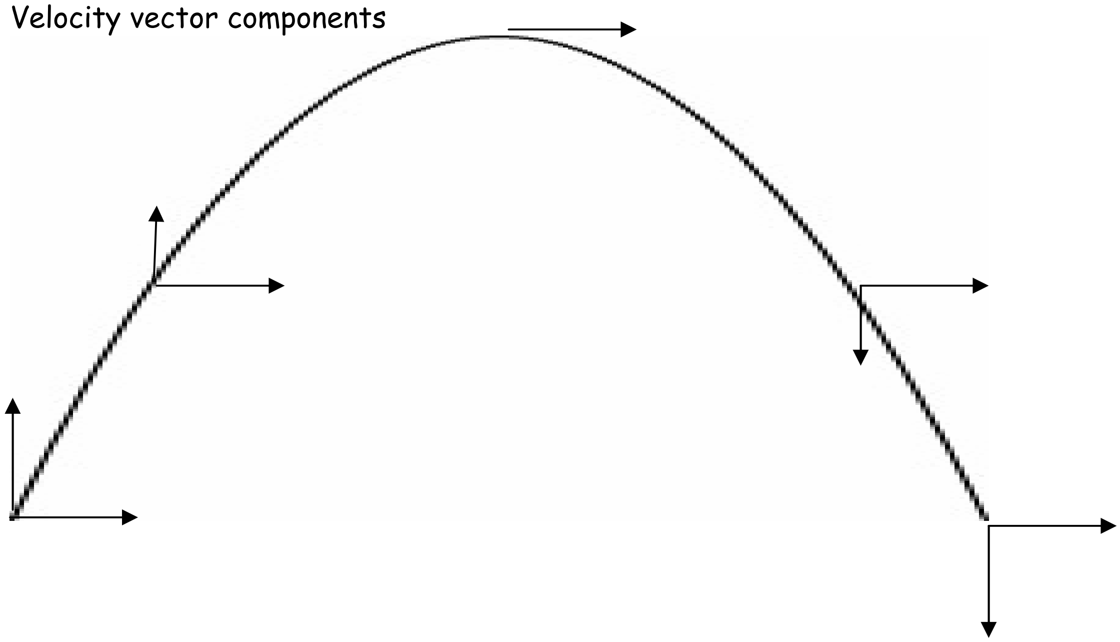
Motion in a Plane/Parabolic Motion

- All motion in a plane problems exist in a Cartesian plane
- Simultaneous motion is present in the *x* and *y* coordinates
- Two sets of kinematic equations are used, one for the *x* direction and one for the *y* direction
- Solutions to the *x* and *y* kinematic equations form the components to solution vectors for displacement and velocity

Convenient Physical Facts/Points of Interest

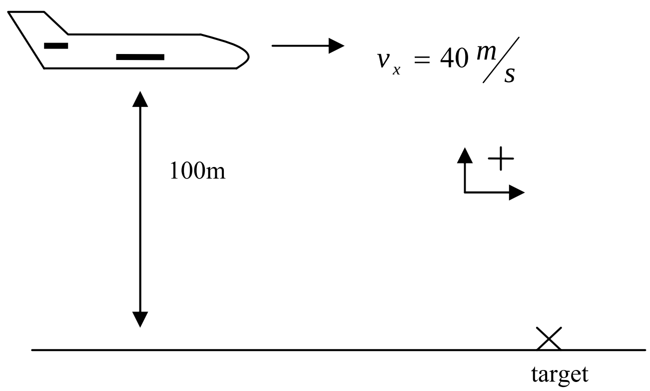
- Acceleration in the *x* direction is always zero so velocity in the *x* direction is always constant and equal to v_0
- Acceleration in the *y* direction is always -9.8m/s^2
- Initial velocity in the *y* direction is zero for a horizontal start
- Velocity in the *y* direction is always zero at the top of the parabola
- In most motion in a plane problems computing time of flight is a good first step
- Remember the vector nature of all equations
- All vectors will have *x* and *y* components: displacement, velocity acceleration
- Remember all of the problem solving techniques from the 1D motion problem set.

Velocity vector components



Example 1

Consider a supply airplane attempting to airdrop a box onto a target marked on the ground as shown below.



At what range from the target should the package be dropped?

Determine time of flight from a height of 100m:

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$-100m = 0 - (4.9m \cdot s^{-2})t^2$$

$$t = 4.5s$$

For a velocity of 40 m/s in the + x direction, what is the range?

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

$$x - x_0 = (+40m \cdot s^{-1})(4.5s) + 0$$

$$x - x_0 = 180m$$

So the package should be released when the plane is 180 meters from the target as long as the plane is in straight and level flight with a constant velocity of 40m/s. What factor is ignored in this problem that would certainly affect the accuracy of this drop?

Example 2

A BASE jumper ascends El Capitan (1200m) in Yosemite Valley CA and in order to BASE jump rather than walk 8 arduous miles down the Yosemite Falls trail. If she leaps horizontally with a velocity of 5 m/s, and due to a technical malfunction is unable to open her parahute until 7 seconds have elapsed, does she live or die?

Given: $y - y_0 = -1200m$

$$a = -9.8\,m/s^2$$

$$t = 7\,s$$

- Try solving for the vertical distance an object in free fall covers in 7 seconds. Note: There is no initial velocity in the y direction.

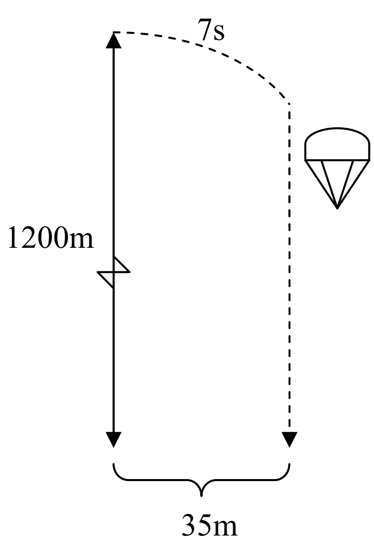
$$y - y_0 = v_{0y}t + \frac{1}{2}at^2$$

$$y - y_0 = (-4.9m/s^2)(7s)^2$$

$$y - y_0 = -240m$$

Since 240m < 1200m she lives.

Where does she land? (Assume she falls vertically once the parachute opens)



Since $v_x = v_{0x}$

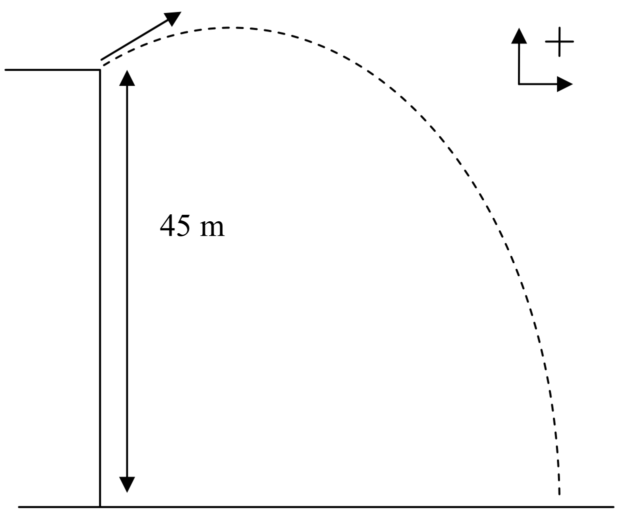
$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

$$= (5m/s)(7s)$$

$= 35m$

Example 3

A stone is thrown from the top of a building upward at an angle of 30° to the horizontal with an initial speed of 20m/s. If the height of the building is 45m, find: time of flight; range; and the velocity of the stone just before it hits the ground.



- Find the components of the initial velocity

$$v_{0x} = v_0 \cos\theta_0 = (20m \cdot s^{-1})(\cos 30°) \quad = +17.3m \cdot s^{-1}$$

$$v_{0y} = v_0 \sin\theta_0 = (20m \cdot s^{-1})(\sin 30°) \quad = +10.0m \cdot s^{-1}$$

- Find the time of flight. Note that this is the same as it would be for a stone tossed straight up at an initial velocity of 10 m/s. Why?

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$-45m = (+10m \cdot s^{-1})(t) - (4.9m \cdot s^{-2})(t^2)$$

$$t = 4.22s$$

- Find the range. In order to do this we use the time of flight (4.22s) and ask ourselves how far in the *x* direction can the stone move at a constant velocity of 17.3 m/s during this time.

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

$$= (17.3m \cdot s^{-1})(4.22s) + 0$$

$$= 73m$$

- What is the velocity of the stone just before striking the ground? Remember that the *x* component of this vector has a constant value that we have already computed (+17.3 m/s). All we have to do is compute the *y* component which we do with our kinematic equations in *y*.

$$v_{fy} = v_{0y} + gt$$

$$= (+10m \cdot s^{-1}) - (9.8m \cdot s^{-2})(4.22s)$$

$$v_{fy} = -31.4m \cdot s^{-1}$$

$$v_{fx} = v_{0x} = +17.3 \, m/s$$

Now resolve the *x* and *y* components into the resultant velocity vector

$$v = \sqrt{v_x^{\ 2} + v_y^{\ 2}} \quad = \quad \sqrt{(17.3)^2 + (-31.4)^2} m \cdot s$$

$$= 35.9m \cdot s^{-1}$$

$$tan\,\theta = \frac{y}{x} = \frac{-31.4m \cdot s^{-1}}{+17.3m \cdot s^{-1}}$$

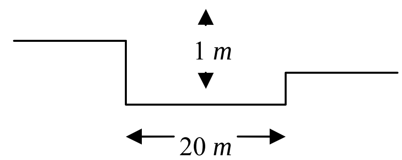
$$\theta = -61°$$

So the velocity just before the stone hits the ground is:

$$\vec{v} = 35.9m \cdot s^{-1} \text{@ } -61°$$

Example 4

A mountain biker approaches a ditch from the left at a speed of 16 m/s. The ditch is 20 m wide and the bank on the opposite side is 1 meter lower. Does the mountain biker make it across?



Find the time of flight for a vertical distance of 1 meter, i.e., the time it takes to drop a distance of 1 meter which places the airborne biker below the opposite lip of the ditch.

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$-1 = 0 - (4.9\, {}^{m}\!/_{s^2})t^2$$

$$t = .452s$$

Now find the range for this time of flight.

Now: $x - x_0 = v_{0x}t$

$$= (16\, {}^{m}\!/_{s})(.452s)$$

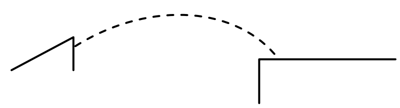
$$= 7.2m$$

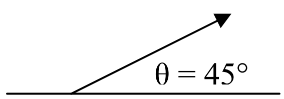
So the mountain biker doesn't clear the ditch.

Example 5

The mountain biker in the previous example is being chased by a bear and really needs to get across the ditch. Unfortunately he can't pedal any faster. Any suggestions?

One could increase the time of flight by launching at an angle by use of a "kicker" or incline. Let's assume that the top of the incline is still 1 meter above the other side of the ditch. It may easily be determined by experimentation that the maximum horizontal range is achieved through a launch angle of 45^0 for most circumstances.





The components of the initial velocity are:

$$v_{0x} = v_0 \cos 45° = 11.3 m \cdot s^{-1}$$

$$v_{0y} = v_0 \sin 45° = 11.3 m \cdot s^{-1}$$

Time of flight at 45° is:

$$y - y_0 = v_{0y} t - \frac{1}{2} g t^2$$

$$-1m = \left(+11.3 m \cdot s^{-1}\right) t - \left(4.9 m \cdot s^{-2}\right) t^2$$

The y time of flight equation is a quadratic in t:

$$\frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

$$a = -4.9$$

$$b = +11.3$$

$$c = 1$$

$$t = \frac{-11.3 \pm \sqrt{(b^2 - 4ac)}}{(2)(-4.9)}$$

$$t = \frac{-11.3 \pm (12.1)}{-9.8}$$

$$t = 2.39s$$

Since $v_x = 11.3$ m/s, in 2.39s the biker travels 27m, so he makes it across.

Example 6

A kicker is capable of booting a football at an angle of θ = 37° with an initial velocity of 20 m/s.

Find:

- Maximum height
- Time of flight
- Range
- Velocity at max. height
- Acceleration at max. height
- Maximum field goal range if bar is 3m tall

First let's find the *x* and *y* components of initial velocity

$$v_{0x} = v_0 \cos\theta \quad = \quad v_0 \cos 37° = 16\,{}^{m}\!/_{s}$$
$$v_{0y} = v_0 \sin\theta \quad = \quad v_0 \sin 37° = 12\,{}^{m}\!/_{s}$$

Does it make sense that the *x* component is larger? Why? Next, noting that maximum height occurs when $v_y = 0$

$$v_{fy} = v_{0y} + a_y t$$

$$\therefore \frac{v_{0y}}{g} = t = \frac{12\,{}^{m}\!/_{s}}{9.8\,{}^{m}\!/_{s^2}} = 1.22s$$

So maximum height occurs 1.22 seconds into the airborne trajectory of the football. To compute the height we simply use this information to compute the vertical displacement (Without any further calculation can you tell what the total time of flight will be given the symmetry of this problem?).

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$= (12\,{}^{m}\!/_{s})(1.22s) - (4.9\,{}^{m}\!/_{s^2})(1.22s)^2$$

$$y - y_0 = +7.35m$$

Maximum height is 7.35 m above the ground.

The total time of flight may be calculated (if you have not already figured it out using the symmetry of the problem) with the knowledge that when $y - y_0 = 0$ the ball is on the ground. This occurs twice, when $t = t_0$ and when $t = t_f$. Hence:

$$y - y_0 = (12\,{}^{m}\!/_{s})t - (4.9\,{}^{m}\!/_{s^2})t^2$$

which is a quadratic in *t*:

$$t_0 = 0$$

$$t_f = 2.45s$$

Now to find the range we use the time of flight and the constant value of the velocity in the *x* direction: Since $v_x = v_{0x}$

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2$$

$$= (16\,{}^{m}\!/_{s})(2.45s)$$

= 39.2*m*

Or use the *range equation*:

$$R = \frac{{v_0}^2 \sin 2\theta_0}{g}$$

$$= \frac{(20\,{}^{m}\!/_{s})^2 \sin 74°}{9.8\,{}^{m}\!/_{s^2}}$$

R = 39.2*m*

To determine the velocity at the apex we exploit the facts that v_x is constant and v_y is zero. Hence:

$$v_x = v_{0x} = 16m/s$$
$$v_y = 0$$
$$\because v = 16m/s$$

To determine the acceleration at apex we exploit the facts that acceleration in the *y* direction is constant and acceleration in the *x* direction is zero. Hence:

$$a_y = -g = -9.8m/s^2$$
$$a_x = 0$$
$$\because a = -9.8m/s^2$$

To determine the maximum field goal range for a 3m high crossbar we must cast the physics problem in mathematical terms. We want to find $x - x_0$ at the same time $y - y_0 = +3m$. We can do this by solving for the time when this relationship is true.

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$+3m = (12 \, {}^{m}\!/_{s})t - (4.9 \, {}^{m}\!/_{s^2})t^2$$

Quadratic: a = -4.9, b = 12, c = -3 → t = 0.29s ↑ up; t = 2.16s ↓ down

So the football is at a height of 3 meters at *t* = 0.29 seconds on its way up and *t* = 2.16 seconds on its way down (which is what we want for maximum range). The next step is to find out how far the ball has traveled from horizontally from the origin in this time. Hence:

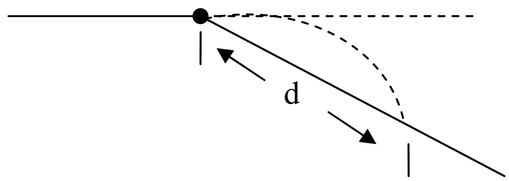
$$x - x_0 = v_{0x}t$$

$$= (16 \, {}^{m}\!/_{s})(2.16s)$$

= 34.6 meters

Example 7

A skier cranking down the ridge at Pebble Creek hits the cat track with a velocity of 15 m/s. If the skier leaves the car track horizontally and if the ground falls away with a slope of -20^0 how long is the skier in the air? What is the skier's velocity upon landing? What does this suggest to you about the desirability of steep landings after if you are going to jump off something on skis or on a dirt or mountain bike?



$v_0 = 15 \, {}^m\!/\!_s$

Slope = -20°

Since the ground falls away, we have to divide the displacement vector, *d*, to the landing into components:

$$\therefore \quad x - x_0 = v_0 t + \frac{1}{2} a t^2$$

$$d\cos\theta = x - x_o \quad + \quad d\sin\theta = y - y_o$$

=> 1. $d\cos\theta = v_{0x} t + \frac{1}{2} a_x t^2$ 2. $d\sin\theta = v_{0y} t - \frac{1}{2} g t^2$

$d\cos\theta = 15 \, {}^m\!/\!_s \, t$ $d\sin\theta = -\frac{1}{2} g t^2$

2 equations, 2 unknowns, *d* and *t*

One method of solution is to divide equation 2 by equation 1, eliminate the variable *d* while solving for *t*:

$$\frac{d\sin\theta}{d\cos\theta} = \frac{-\frac{1}{2}gt^2}{15\,{}^{m}\!/\!{}_{s}\,t} \Rightarrow \frac{-\frac{1}{2}(9.8\,{}^{m}\!/\!{}_{s^2})t}{15\,{}^{m}\!/\!{}_{s}} = \tan 20°$$

$$t = 1.11s$$

Next we solve for the velocity components on landing:

$$v_{fx} = v_{0x} = 15\,{}^{m}\!/\!{}_{s} \qquad v_{fy} = -(9.8\,{}^{m}\!/\!{}_{s^2})(1.11s) \approx -11\,{}^{m}\!/\!{}_{s} \quad \Rightarrow \quad \tan^{-1}\left(\frac{-11\,{}^{m}\!/\!{}_{s}}{+15\,{}^{m}\!/\!{}_{s}}\right) = -36°$$

$$v = \sqrt{(15\,{}^{m}\!/\!{}_{s})^2 + (11\,{}^{m}\!/\!{}_{s})^2} = 18.5\,{}^{m}\!/\!{}_{s}$$

So $\vec{v} = 18.5\,{}^{m}\!/\!{}_{s}$ at an angle of 36° below horizontal

The steeper the landing, in general, the closer it comes to aligning with the velocity vector. This insures soft (albeit high speed) landings because there is little immediate change in acceleration due to a direction change on landing.

Example 8

A boat lies at anchor 100 meters offshore. I launch a water balloon from the beach at this boat with an initial velocity of 50 m/s. At what angle must I launch the balloon in order to strike the boat?

Given: $v_0 = 50 \,{}^{m}\!/_{s}$

$x - x_0 = 100m$

Find: θ

The challenge here is to solve for the launch angle – information that has been supplied with problems we've looked at so far. Since we don't have an equation that can be solved directly for θ (without also having to solve for other variables, time, velocity, etc.) we will have to derive what we need from our list of kinematic equations.

Using our stock kinematic equations and substituting for known quantities:

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 \rightarrow 0 = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \Rightarrow \frac{2v_0 \sin\theta}{g} = t$$

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2 \rightarrow x - x_0 = (v_0 \cos\theta)t \Rightarrow \frac{x - x_0}{v_0 \cos\theta} = t$$

If these equations are set equal to each other t is eliminated and we are left to solve for θ in terms of initial velocity (v_o) and displacement ($x - x_o$) both which are given.

$$\frac{2v_0 \sin\theta}{g} = \frac{x - x_0}{v_0 \cos\theta} \therefore 2\sin\theta\cos\theta = \frac{(x - x_0)g}{{v_0}^2}$$

$$\sin 2\theta = \frac{(100m)(9.8m \cdot s^{-2})}{(50m \cdot s^{-1})^2}$$

$2\theta = \sin^{-1} 0.392 \therefore \theta \approx 11.54°$ So I'd need to launch the balloon upwards at an angle of about 11.5 degrees.