

Magnetostatics

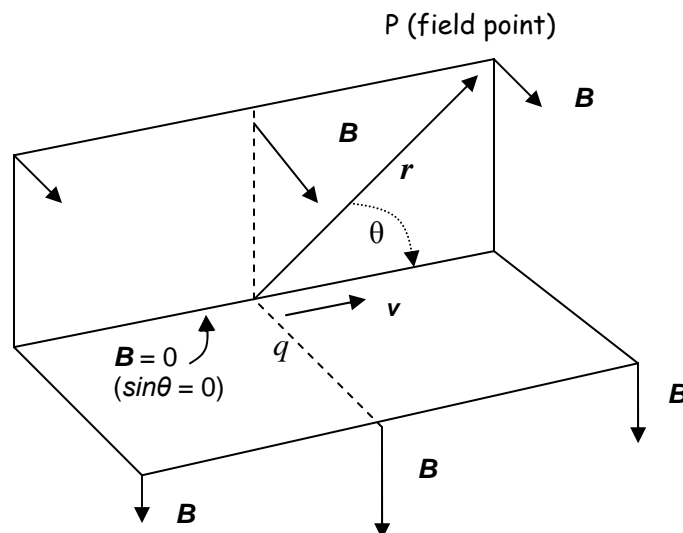
We have previously considered the effect of an *in situ* magnetic field on a charged particle. We now want to consider more closely the sources of these magnetic fields

All magnetic fields are due to current loops:

- The current may be the continuous current in a conductor
- Or the current may be the effective current of a charged particle in motion

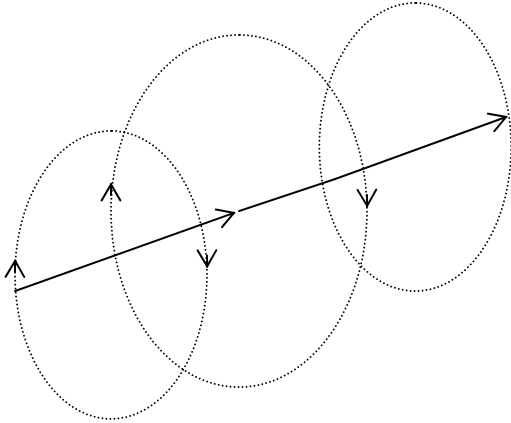
Consider a point charge, q , moving with velocity, v , in a straight line as shown below. it may be shown that:

- B is proportional to q and $\frac{1}{r^2}$
- B is \perp to the plane containing the particle's trajectory
- B is proportional to $\sin\theta$ between v and r
- The magnitude of B is given by: $B = k' \frac{qv \sin \theta}{r^2}$ where k' is a constant of proportionality not related to k .
- An electric field is also produced (not shown) as a result of the electric charge.



- $B = 0$ along the path of the charge
- B is at its maximum value at points in the plane \perp to v and along the path of the charge where $\sin\theta = 1$ ($\theta = 90^\circ$)

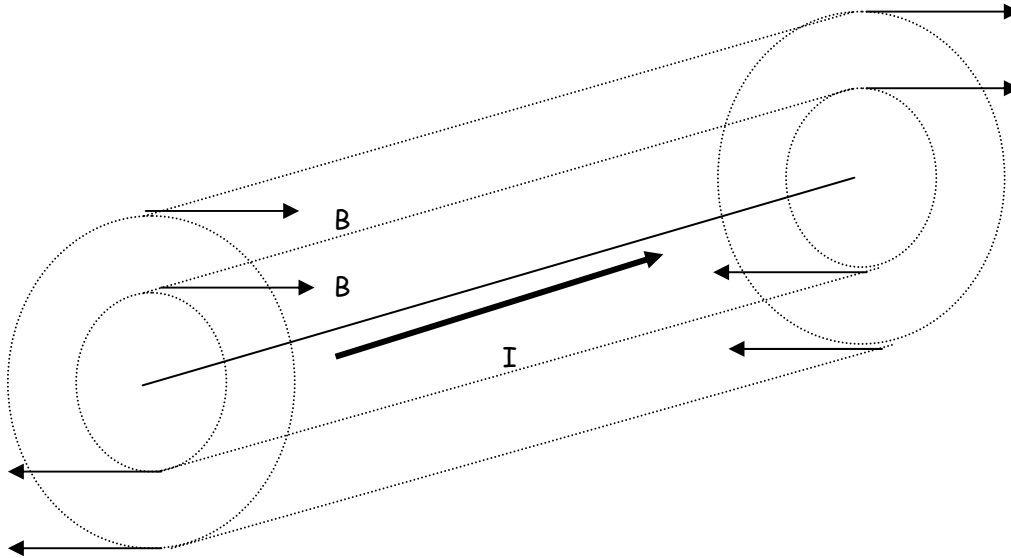
- Define $\hat{r} = \frac{\vec{r}}{r}$ so that $\vec{B} = k' \frac{q\vec{v} \times \hat{r}}{r^2}$
- The magnetic field lines are fundamentally different from \vec{E} field lines. \vec{E} lines radiate outward from charges. \vec{B} lines encircle the charges (have no beginning or end).



- RHR gives direction of magnetic field lines. One points their thumb in the direction of \vec{v} curls their fingers towards their palm and their fingers point in the direction of \vec{B}
- $k' = \frac{\mu_0}{4\pi}$ where $\mu_0 = 4\pi \times 10^{-7} \text{ N} \cdot \text{A}^{-2}$ and is the *permeability* of free space
- $B = \frac{\mu_0}{4\pi} \frac{qv \sin \theta}{r^2}$ or $\vec{B} = k' \frac{q\vec{v} \times \hat{r}}{r^2}$
- $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

To perform the same analysis on a current carrying conductor one uses the principle of superposition to add the magnetic fields of all the moving charges together.

For a long, straight conductor the magnetic field encircling the conductor consists of a series of concentric cylinders.



Magnetic Flux

Electric and magnetic flux are defined in the same manner.

$$\Phi_M = \int \vec{B} \cdot d\vec{a}$$

- Magnetic field lines have no beginning or end, unlike electric field lines.
- When using Gauss's Law to evaluate \vec{E} fields we exploit the fact that the number of field lines leaving a Gaussian surface is related to the amount of charge enclosed within the surface. The situation is quite different with \vec{B} fields which are continuous loops. For any surface that encloses a current the number of magnetic field lines that enter and leave the surface must be the same so the net magnetic flux must be zero. Hence:

$$\oint \vec{B} \cdot d\vec{a} = 0$$

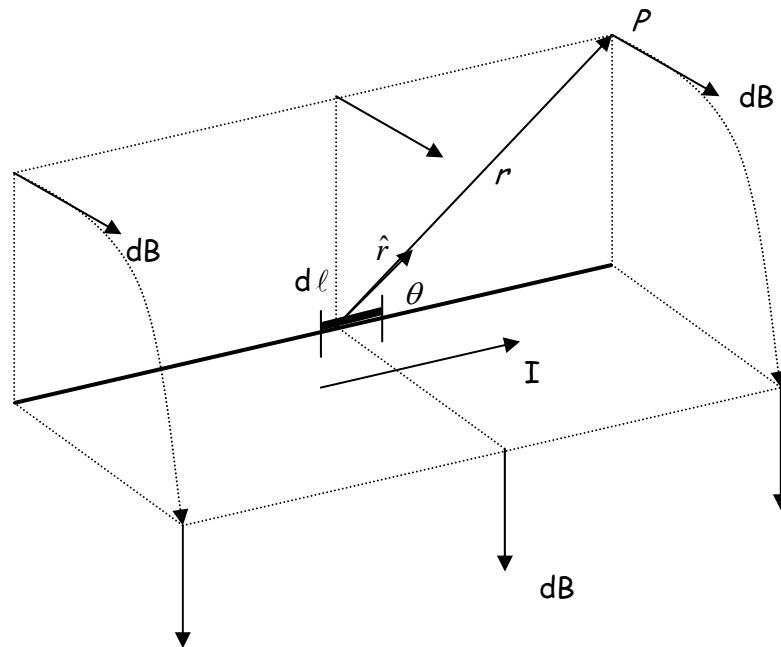
Gauss Law of Magnetism

- There are no magnetic monopoles.
- Only magnetic dipoles have been found

Computing Magnetic Fields Due to Current Carrying Elements

Biot-Savart

The magnetic field produced at a field point P by the current in a conductor is the vector sum of the fields due to all of the moving charges within the conductor.



- $d\vec{B}$ is \perp $d\vec{\ell}$ and to \hat{r}
- The magnitude of $d\vec{B}$ is inversely proportional to r^2 , proportional to the current I and the length $d\ell$ of the element, and proportional to $\sin\theta$ where θ is the angle between $d\ell$ and r .
- The volume of the segment of conductor $d\ell$ is $A \times d\ell$.
- Recall that if there are n charges of magnitude q in the segment $d\ell$, the current in the segment $d\ell$ may be defined as $I = nqv_d A$ where v_d is the drift velocity of the charges.
- The total moving charge in the element is $dQ = nqAd\ell$
- Recall $B = \frac{\mu_0}{4\pi} \frac{qv \sin\theta}{r^2}$ or $\vec{B} = k' \frac{q\vec{v} \times \hat{r}}{r^2}$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{dQv \sin\theta}{r^2} = \frac{\mu_0}{4\pi} \frac{nqAd\ell v \sin\theta}{r^2} = \frac{\mu_0}{4\pi} \frac{Id\ell \sin\theta}{r^2} \rightarrow d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{\ell} \times \hat{r}}{r^2}$$

To find the total \mathbf{B} field we integrate:

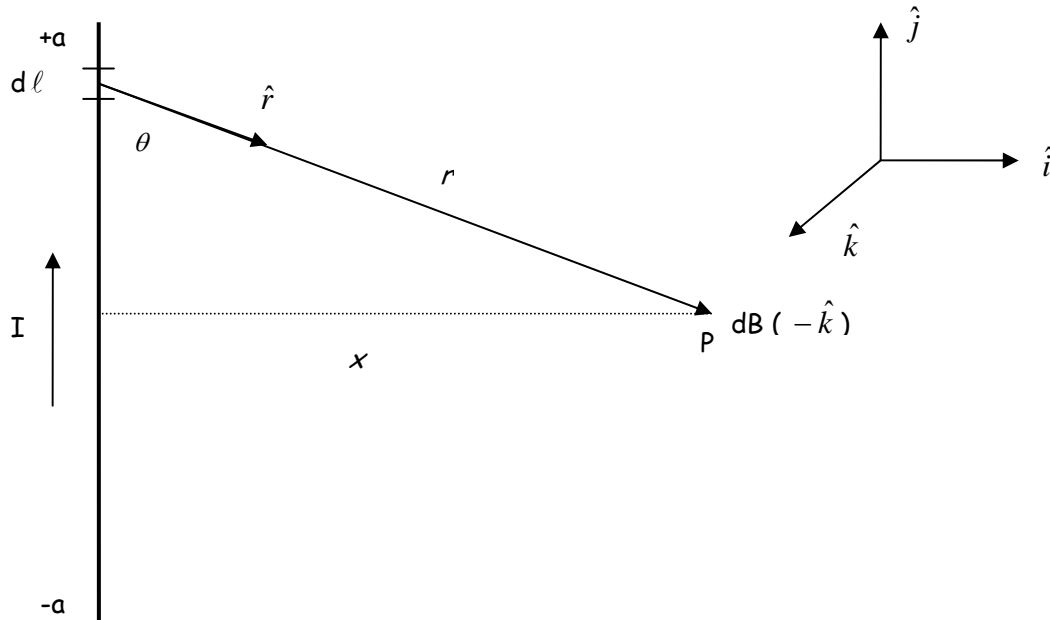
$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{\ell} \times \hat{r}}{r^2}$$

This is known as the Biot-Savart law.

- Notice that the integrand not only contains vectors but is, as a result of the cross product, a vector quantity.
- Biot-Savart parallels Coulomb's Law for electrostatics, i.e., charge(s) in motion create magnetic fields that are proportional to charge, distance, and velocity while static charges create electric fields that are proportional to charge and distance.
- Both fields vary in magnitude as $\frac{1}{r^2}$
- Utility of Biot-Savart is limited to geometries where the integrand is easily evaluated.

Applications of Biot-Savart

Magnetic Field due to a long, straight wire.



The RHR indicates that dB is into the plane of the page (\times).

$$d\vec{\ell} \times \hat{r} = dy(1) \sin \theta (-\hat{k})$$

$$dB = \frac{\mu_0 I}{4\pi} \frac{dy \sin \theta}{r^2} (-\hat{k})$$

Notice that we have three variables that we will need to relate before integration: θ , y , r .

$$r = (x^2 + y^2)^{1/2} \text{ and } \sin \theta = \frac{y}{r} = \frac{y}{(x^2 + y^2)^{1/2}}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-a}^{+a} \frac{y dy}{(x^2 + y^2)^{3/2}} (-\hat{k})$$

Evaluating the integral:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2 + a^2}} (-\hat{k})$$

Alternatively:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{Id\vec{\ell} \times \hat{r}}{r^2}$$

Note: $d\vec{\ell} = dy\hat{j}$, $\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} - y\hat{j}}{\sqrt{x^2 + y^2}}$ The term in the numerator represents the components of vector r .

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-a}^{+a} \frac{dy\hat{j} \times (x\hat{i} - y\hat{j})}{(x^2 + y^2)^{3/2}}$$

Evaluating the cross product:

$$\begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 0 & dy & 0 & 0 & dy \\ x & -y & 0 & x & -y \end{array} = -x dy (\hat{k})$$

Transferring the minus sign to the unit vector:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-a}^{+a} \frac{x dy}{(x^2 + y^2)^{3/2}} (-\hat{k})$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2 + a^2}} (-\hat{k})$$

Let's see what happens if we make this current carrying conductor very long.

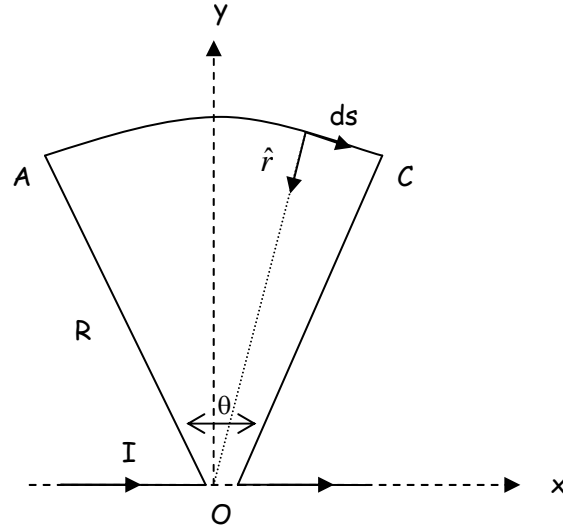
$$B = \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2 + a^2}} = \frac{\mu_0}{4\pi} \frac{2a}{x\sqrt{\frac{x^2}{a^2} + 1}}$$

If $a \gg x$ (or if $a \rightarrow \infty$) then: $B = \frac{\mu_0 I}{2\pi x}$

In terms of a general coordinate: $B = \frac{\mu_0 I}{2\pi r}$. This is the magnetic field for any long straight conductor. Notice that the B field is directly proportional to the current and inversely proportional to the distance from the conductor.

Magnetic Field of a Current Loop

Consider the current loop shown at right. What is the magnetic field at point O due to the current in the loop through segments OA , AC and CO ?



Notice that ds is respectively antiparallel and parallel to \hat{r} along segments OA and CO and the cross product in Biot-Savart vanishes. The magnetic field at point O due to current in these segments is zero.

Each segment ds along the curved element from A to C is the same distance (R) from O . Notice also that ds along this path is everywhere \perp to \hat{r} . Hence:

$$dB = \frac{\mu_0 I}{4\pi R^2} ds \rightarrow B = \frac{\mu_0 I}{4\pi R^2} \int ds$$

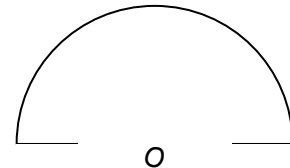
This expression is easily evaluated since R and I are constants and ds is the arc length from A to C (in radians). Noting that $s = R\theta$.

$$B = \frac{\mu_0 I}{4\pi R^2} \int ds = \frac{\mu_0 I s}{4\pi R^2} = \frac{\mu_0 I \theta}{4\pi R}$$

The RHR gives the direction as into the page.

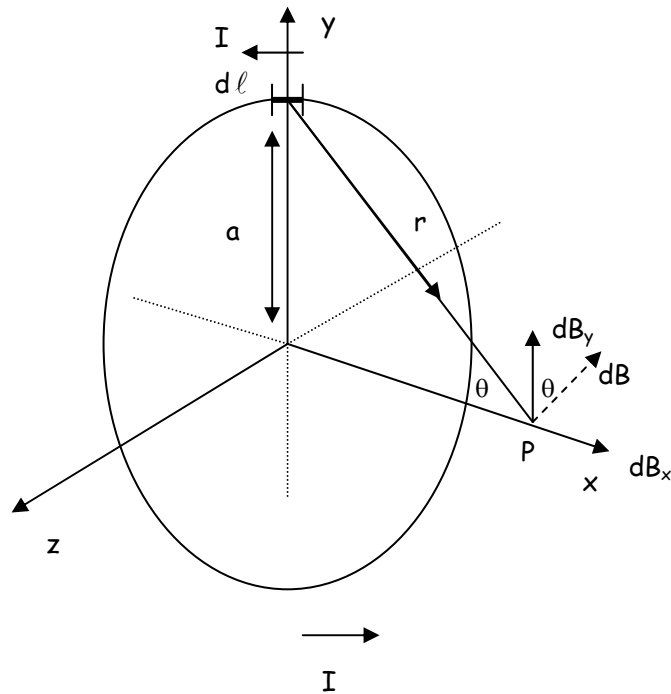
This expression will work for any curved segment of constant radius. Notice that if the distance from A to C were π radians.

$$B = \frac{\mu_0 I}{4\pi R^2} \int_0^\pi ds = \frac{\mu_0 I}{4R}$$



Magnetic Field of a Circular Loop.

Consider a current flowing ccl through a loop in the y-z plane of radius a centered on the x-axis as shown at right. What is the magnetic field at a point P located some distance x from the plane of the loop along the x-axis?



The RHR applied at $d\vec{\ell}$ indicates that $d\vec{B}$ lies in the x-y plane.

- $r = \sqrt{x^2 + a^2}$
- $d\vec{\ell} \perp r$
- $\sin \theta = \frac{a}{\sqrt{x^2 + a^2}}$
- $\cos \theta = \frac{x}{\sqrt{x^2 + a^2}}$

Biot-Savart: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{\ell} \times \hat{r}}{r^2}$ or $dB = \frac{\mu_0 I d\ell \sin \theta}{4\pi r^2}$

Using the first expression and substituting: $dB = \frac{\mu_0 I}{4\pi} \frac{d\ell}{x^2 + a^2}$

$$dB_x = dB \sin \theta = \frac{\mu_0 I}{4\pi} \left(\frac{d\ell}{x^2 + a^2} \right) \left(\frac{a}{(x^2 + a^2)^{1/2}} \right)$$

The components of $d\vec{B}$ are:

$$dB_y = dB \cos \theta = \frac{\mu_0 I}{4\pi} \left(\frac{d\ell}{x^2 + a^2} \right) \left(\frac{x}{(x^2 + a^2)^{1/2}} \right)$$

Notice that due to symmetry about the x-axis $\sum dB_y = 0$ so the B field points in the x direction and all we have to concern ourselves with is dB_x .

$$dB = dB_x = dB \sin \theta = \frac{\mu_0 I}{4\pi} \left(\frac{d\ell}{x^2 + a^2} \right) \left(\frac{a}{(x^2 + a^2)^{3/2}} \right) \rightarrow B = \frac{\mu_0 I}{4\pi} \int \frac{ad\ell}{(x^2 + a^2)^{3/2}}$$

Inspection of the integrand reveals that a and x are constant so:

$$B = \frac{\mu_0 I a}{4\pi (x^2 + a^2)^{3/2}} \int d\ell$$

Integration of $d\ell$ all around the current loop yields $2\pi a$.

$$B = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}}$$

This analysis yields a vector with the aid of the RHR (pointing along the x-axis). We could, alternatively evaluate the integral using the second, vector version of Biot-Savart:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{\ell} \times \hat{r}}{r^2}$$

In terms of vectors: $d\ell = d\ell \hat{k}$, $\vec{r} = x\hat{i} - a\hat{j}$, and, as always $\hat{r} = \frac{\vec{r}}{r}$, hence:

$$\frac{d\vec{\ell} \times \hat{r}}{r^2} = \frac{d\vec{\ell} \times \vec{r}}{r^3} = \frac{d\vec{\ell} \hat{k} \times (x\hat{i} - a\hat{j})}{(x^2 + a^2)^{3/2}} = \frac{xd\ell \hat{j} + ad\ell \hat{i}}{(x^2 + a^2)^{3/2}} \leftarrow \begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 0 & 0 & d\ell & 0 & 0 \\ x & -a & 0 & x & -a \end{array} = xd\ell \hat{j} + ad\ell \hat{i}$$

The $xd\ell \hat{j}$ term is the y component of $d\vec{B}$ which sums to zero as we integrate around the loop. Hence:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{ad\ell \hat{i}}{(x^2 + a^2)^{3/2}} = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \hat{i}$$

Let's examine our result a little more closely. Notice that at the center of the loop ($x = 0$) the expression reduces to:

$$B = \frac{\mu_0 I}{2a}$$

If some multiple n circular loops are involved the B field is:

$$B = n \frac{\mu_0 I}{2a}$$

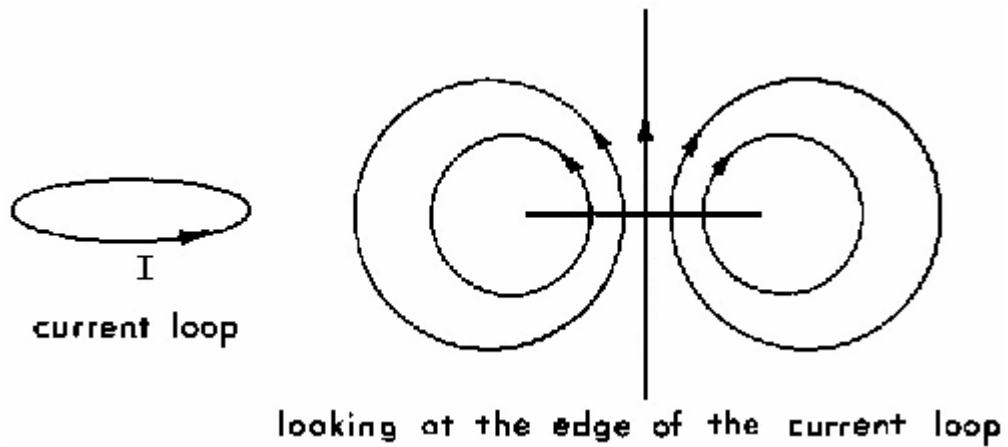


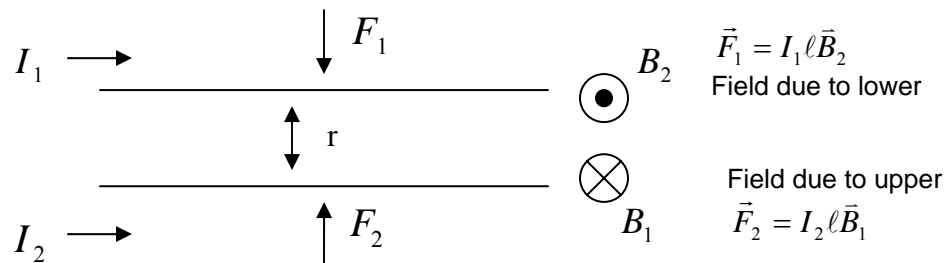
Illustration courtesy of <http://www.brooks.af.mil>

For additional analysis of the field due to current loops have a look at:

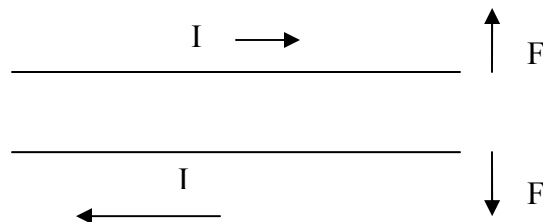
<http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/curloo.html>

Magnetic Force between Parallel Conductors

We wish to investigate the case of two parallel current carrying conductors and the effects that the magnetic fields of each have on the other. We will assume that the conductors are long and relatively close to each other.



- The magnetic field created by the lower conductor is: $B_2 = \frac{\mu_0 I_2}{2\pi r}$ (magnetic field around a wire)
- The force on the upper conductor due to the lower is: $F_1 = I_1 \ell B_2 = \frac{\mu_0 I_1 I_2 \ell}{2\pi r}$
- Since the length of these conductors is not specified, it's best to write the force in terms of force per unit length: $\frac{F}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi r}$
- RHR shows that the force on the upper conductor due to the \mathbf{B} field from the lower is directed downward.
- Newton III (and simple symmetry) says that the lower conductor experiences an equal and opposite force upward due to the field from the upper conductor.
- The conductors are attracted to each other.
- If the current in either of the wires is reversed:



The force between the two is repulsive.

Pinch Effect:

- Mutual forces exist not only between parallel conductors but also between longitudinal elements in the same conductor.
- If the conductor is a liquid or ionized gas (plasma) these forces result in the constriction of the conductor, as if some external pressure were being applied.
- This constriction produces a high temperature in the plasma due to the high pressure.
- A technique used to induce nuclear fusion.

The Ampere Defined

The unvarying current which, if present in each of the two parallel conductors of infinite length and one meter apart in empty space, causes each conductor to experience a force of 2×10^{-7} N/m of length.

Ampere's Law

An alternative to Biot-Savart for computing magnetic fields.

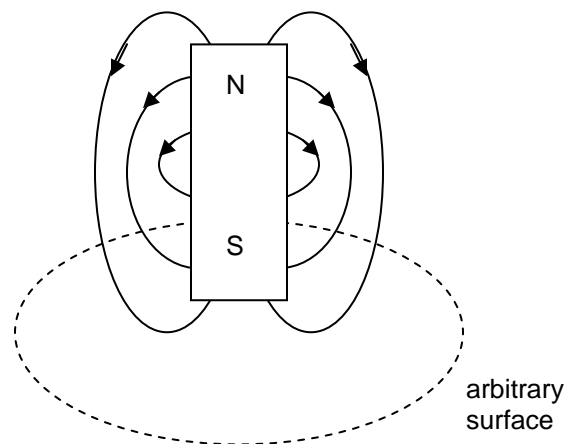
Analogous to using Gauss' Law for computing \mathbf{E} fields

Ampere's Law relates tangential components of the \mathbf{B} field on a closed path surrounding the current to the current itself.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} \quad \text{Ampere's Law}$$

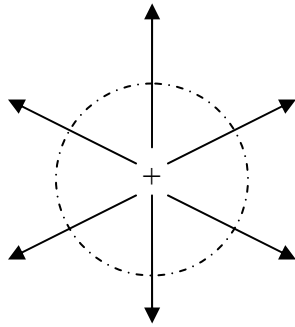
$$\oint \vec{B} \cdot d\vec{a} = 0 \quad \text{Gauss' Law for magnetic fields (true but not particularly useful)}$$

Consider



- $\oint \vec{B} \cdot d\vec{a} = 0$ or $\nabla \cdot \vec{B} = 0$
- The net magnetic flux through any closed surface is zero.
- *Gauss Law for magnetism implies that there are no magnetic monopoles and no isolated magnetic poles.*
- There are no "point sources" for magnetic fields.
- Magnetic field lines are endless or extend infinitely into space.

In Gauss Law for electrostatics the number of lines crossing a Gaussian surface was proportional to the charge enclosed within the surface.



$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \text{ or } \nabla \cdot \vec{E} = \frac{q_{enc}}{\epsilon_0}$$

This works because electric field lines diverge away from their source. Magnetic field lines do not (zero divergence) so there is no Gauss's Law for \mathbf{B} fields.

Ampere's Law

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

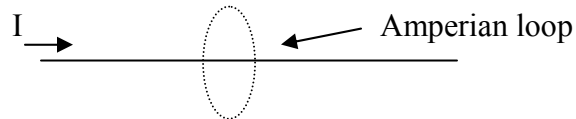
To apply Ampere's law we need to merely surround the current carrying element with an artificial "Amperian loop" of length ℓ that meets the following requirements

- There are no \mathbf{B} field lines perpendicular to the loop (if there are they do not contribute to the total \mathbf{B} field)
- The \mathbf{B} field lines that do contribute are all parallel to the loop
- Currents must be steady (in order to produce a constant \mathbf{B} field)
- Need a high degree of symmetry if the application is to be easy

Verification of Ampere's Law

Consider the magnetic field created by a long, straight wire (we will consider wires to be long conductors of negligible diameter).

Let's construct an Amperian loop that meets the requirements we've discussed, i.e., one that fits the symmetry and encloses the current.



From Biot-Savart we know that the magnetic field for a long straight wire is:

$$B = \frac{\mu_0 I}{2\pi r}$$

and that this \mathbf{B} field is tangent to the Amperian loop at each point. Applying Ampere's law (with \mathbf{B} known):

$$\oint \vec{B} \cdot d\vec{\ell} = B \oint d\ell = \frac{\mu_0 I}{2\pi r} \oint d\ell$$

In this case $d\ell = 2\pi r \rightarrow \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$

So Ampere's Law is confirmed.

Ampere's law works for any closed path. The only difficulty is in evaluating the integrand $\oint d\ell$ for anything other than simple geometries.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} \quad \text{or} \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

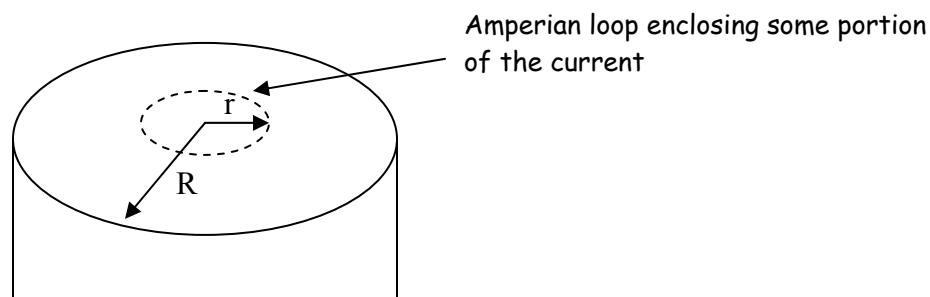
Applications of Ampere's Law

Solid Conducting Cylinder

Unlike a wire a cylinder has some non negligible diameter. We'll look at the magnetic field due to currents inside and outside the conductor.

As we know from electrostatics charges prefer to reside on the surfaces of conductors. Currents also tend to flow along the surfaces of conductors but some current flows beneath the surface and this cannot be ignored for a large diameter cylinder.

To make the calculations less burdensome, assume a uniform current within the conductor.



$$\text{Ampere's Law: } \oint \vec{B} \cdot d\vec{\ell} = B2\pi r = \mu_0 I_{enc}$$

We need to express I_{enc} as a function of the radius of the conductor. We can do this with current density. Recall:

$$\text{Current Density: } J = \frac{I}{\pi R^2} \quad \left(\frac{\text{current}}{\text{area}} \right)$$

The current within the Amperian loop is the product of uniform current density and the area enclosed by the Amperian loop.

$$I_r = J(\pi r^2) = \frac{I\pi r^2}{\pi R^2}$$

$$\text{And since } I_{enc} = \frac{B2\pi r}{\mu_0}$$

$$2\pi rB = \frac{\mu_0 I r^2}{R^2}$$

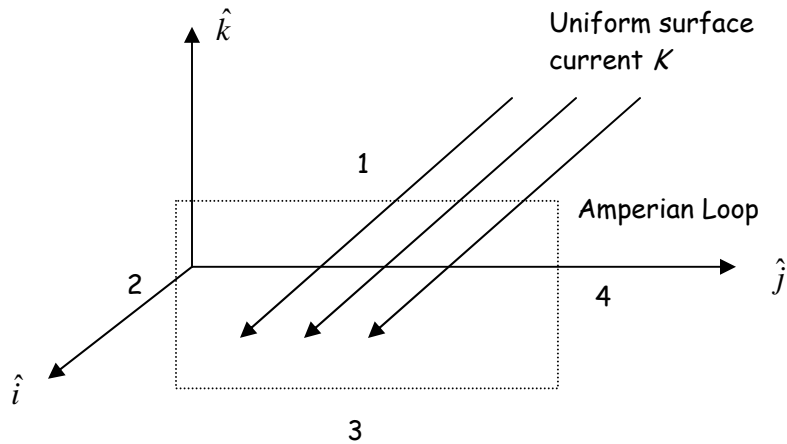
$$B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$

If $r = R$, $B = \frac{\mu_0 I}{2\pi r}$ in agreement with the previous result for any long straight conductor

Sheet of Current

Consider a uniform surface current in the $+x$ direction, i.e., a "sheet" of current with a uniform surface current K .

We must apply Ampere's law piecewise because our Amperian loop, which otherwise meets all of the needed criteria, is discontinuous.



$$\oint \vec{B} \cdot d\vec{\ell} = B\ell = \mu_0 I_{enc}$$

- 1) $B\ell = \mu_0 I_{enc}$
- 2) $\ell \rightarrow 0 \therefore B = 0$
- 3) $B\ell = \mu_0 I_{enc}$
- 4) $\ell \rightarrow 0 \therefore B = 0$

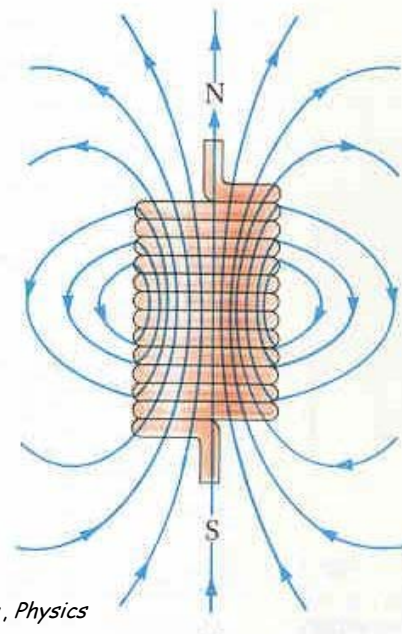
The sum of these is $2B\ell = \mu_0 K\ell \rightarrow B = \frac{\mu_0}{2} K$

RHR gives directions: in the $-\hat{j}$ direction above the sheet and in the $+\hat{j}$ direction below the sheet.

A Solenoid

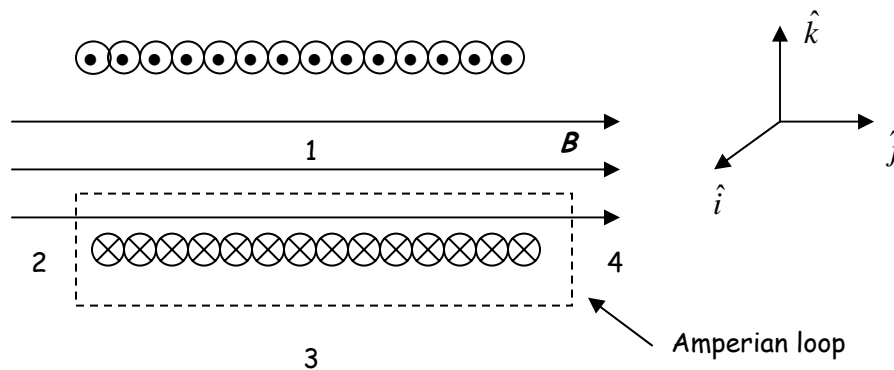
A solenoid is a long insulating tube with a wire conductor wrapped around it. A schematic cross section of such a device is shown below.

Notice that the current in the diagram below flows around the wire windings in such a manner that it goes into the plane of the page at the bottom of the loop and out at the top.



Serway, Physics

Inside the ideal solenoid below the \mathbf{B} field is uniform and directed in the $+\hat{j}$ direction. If the windings are closely spaced and the length of the solenoid is long compared with its radius, the \mathbf{B} field inside may be considered uniform.



$$\oint \vec{B} \cdot d\vec{\ell} = B\ell = \mu_0 I_{enc}$$

$$1) \quad B \parallel d\ell \quad B d\ell = \mu_0 I_{enc}$$

$$2) \quad B \perp d\ell \quad B d\ell = 0$$

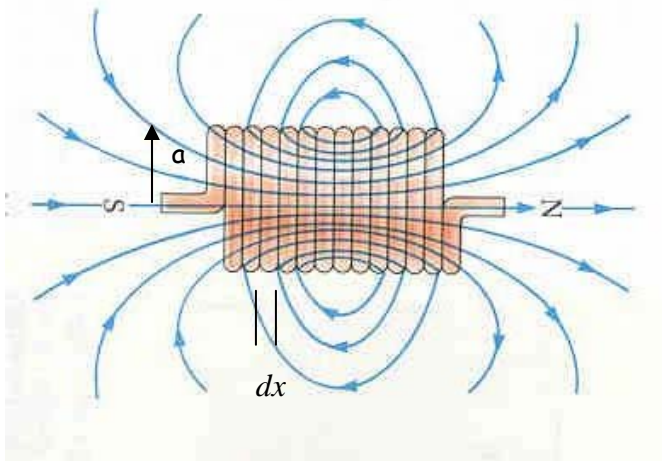
$$3) \quad B = 0 \quad B d\ell = 0$$

$$4) \quad B \perp d\ell \quad B d\ell = 0$$

$$B\ell = \mu_0 I \rightarrow B = \frac{\mu_0 I}{\ell}$$

$$\text{For } N \text{ turns: } B = \frac{N\mu_0 I}{\ell}$$

Sometimes this is expressed in terms of turns per unit length ($n = N/\ell$): $B = n\mu_0 I$



Now consider a "real" solenoid, i.e. one that is not extremely long compared to its radius.

We wish to compute the B field along the axis of the solenoid.

From Biot-Savart: $B = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}}$ for a single circular loop.

$$dB_x = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \frac{N}{\ell} dx$$

$$B_x = \frac{\mu_0 I a^2 N}{2\ell} \int_0^x \frac{dx}{(x^2 + a^2)^{3/2}}$$

$$B_x = \frac{\mu_0 I x N}{2\ell \sqrt{x^2 + a^2}}$$

Notice that if $x \gg a$:

$$B_x = \frac{\mu_0 I N}{2\ell}$$

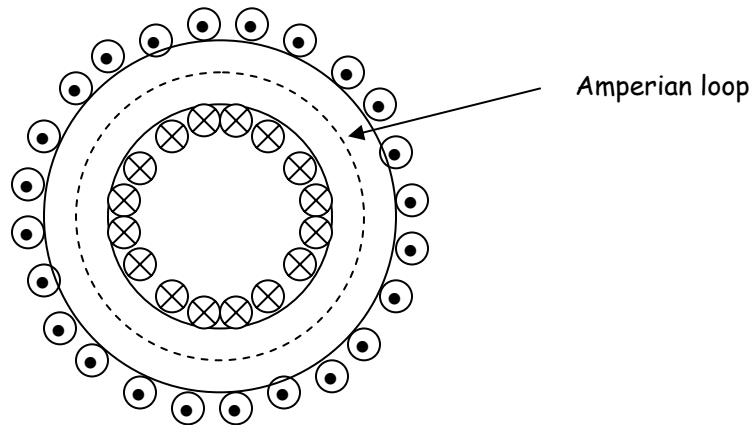
At the midpoint of the solenoid, $\ell/2$ where the most uniform field is found:

$$B_x = \mu_0 n I$$

in agreement with the previous result. At each end $\ell = \ell$ and:

$$B_x = \frac{\mu_0 n I}{2}$$

Torodial Solenoid



$$\oint \vec{B} \cdot d\vec{\ell} = B2\pi r = \mu_0 N I_{enc}$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

Displacement Currents - Maxwell's "Fix" to Ampere's Law

A situation that frequently arises in electrical circuits is one in which electrical current flow is replaced by a changing electric flux, e.g., across the plates of a capacitor that is charging or discharging.

Maxwell postulated that this change in electric flux is equivalent to the "conduction" current that flows elsewhere in the circuit and modified Ampere's Law to account for both types of currents.

$$\text{Conduction current: } I_C \equiv \frac{dQ}{dt}$$

$$\text{Displacement current: } I_D \equiv \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{B} \cdot d\vec{\ell} = B2\pi r = \mu_0 I_{enc} = \mu_0 (I_C + I_D) = \mu_0 I_C + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$