Gravity

Newton's Law of Gravity

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$

- Inverse square
- Conservative
- Always attractive
- *G* = $6.672 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$ (derived empirically)

The magnitude of the force of attraction between any two massive objects is

$\vec{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{r}$ **or** $\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\hat{r}$

G is commonly measured with a device known as a torsion balance.

Courtesy of Pasco

Weight, surface gravity, and the Gravitational Force

Weight and mass are not the same thing. Weight is the effect of mass in Earth's gravitational field.

$w = mg = G\frac{M_e m}{R_e^2} \therefore g = G\frac{M_e}{R_e^2}$ for any body near the earth's surface

On Earth these values yield:

$$g_{earth} = \frac{\left(6.7 \times 10^{-11} m^3 \cdot s^{-2} \cdot kg^{-1}\right)\left(6.0 \times 10^{24} kg\right)}{\left(6.4 \times 10^6 m\right)^2} = 9.8 m \cdot s^2$$

For an 80kg (176 lb) person:

$$F = ma = (80kg)\left(9.8m \cdot s^2\right) = 784 kg \cdot m \cdot s^{-2} = 784N$$

So the gravity exerts a force of 784 Newtons on an 80kg person on the surface of the earth.

At a distance, *h*, above the earth's surface: $g_h = G\frac{M_e}{\left(R_e + h\right)^2}$

Escape Velocity

- Escape velocity is the minimum speed required to escape an object's gravity well
- Independent of mass! A molecule and a spaceship have the same escape velocity
- Escape velocity or, more properly, escape speed is the speed that an object must attain in order to completely escape the gravity of a planet or star. For any roughly spherical object escape speed is:

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

On earth this is:

$$v_{esc} = \sqrt{\frac{2\left(6.7\times10^{-11}m^3\cdot s^{-2}\cdot kg^{-1}\right)\left(6.0\times10^{24}kg\right)}{\left(6.4\times10^{6}m\right)}} = 11200m\cdot s \approx 25{,}000mph$$

Exploring the Solar System with Probes

If we ignore the effects of drag a ballistic rocket launch from a gravitational potential well is a conservative process such that:

$$TE = \frac{1}{2}mv^2 + mgh = \text{constant}$$

the projectile rises to a maximum height:

$$h = \frac{v^2}{2g}$$

This result assumes a constant value of g and a constant force, F_G. For altitudes greater about 1 Earth radius ($h \geq R_\oplus$) this is no longer true. In this case:

$$TE = \frac{1}{2}mv^2 - G\frac{Mm}{r} = \text{constant}$$

if we employ this expression at ground level ($R_\oplus$) and at maximum height ($R_\oplus + h$):

$$\frac{1}{2}mv^2 - G\frac{M_\oplus m}{R_\oplus} = -G\frac{M_\oplus m}{R_\oplus + h}$$

or

$$h = R_{\oplus}\left[\frac{\left(\frac{v^2 R_{\oplus}}{2GM_{\oplus}}\right)}{\left(1-\frac{v^2 R_{\oplus}}{2GM_{\oplus}}\right)}\right] = \left(\frac{v^2}{2g}\right)\frac{R_{\oplus}}{\left[R_{\oplus} - \left(\frac{v^2}{2g}\right)\right]}$$

Note that when $v^2/2g \ll R_{\oplus}$ ($h \ll R_{\oplus}$) the expression reduces to $h = v^2/2g$.

Note that when $v^2/2g = R_{\oplus}$, $v = \sqrt{2gR_{\oplus}}$ (which is 11.2 km/s or about 25,000 mph) $h = \infty$ infinity and the projectile escapes.

Launching a Rocket into Space

- Rocket launches employ conservation of momentum.
- The momentum of light fuel gasses escaping at high speeds propels the heavier rocket forward.
- At some time t the momentum of such a rocket and fuel is: $p = (M + \Delta m)v$.
- At some time Δt later the rocket has ejected a mass of fuel Δm and its velocity has increased by $v + \Delta v$.
- If the gas is ejected with some constant velocity v_e relative to the rocket it's speed in the frame of reference of a stationary observer is $v - v_e$
- Conservation of linear momentum:

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

or

$$M\Delta v = \Delta m(v_e)$$

Taking the limit as $\Delta t \rightarrow 0$: $\Delta v \rightarrow dv$, $\Delta m \rightarrow dm$ and noting that an increase in exhaust mass corresponds to a decrease in rocket mass ($dm = -dM$):

$$Mdv = -v_e dm$$

Noting that as the rocket gains speed it loses mass:

$$\int_{v_i}^{v_f} dv = v_e \int_{M_i}^{M_f} \frac{dm}{M}$$

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

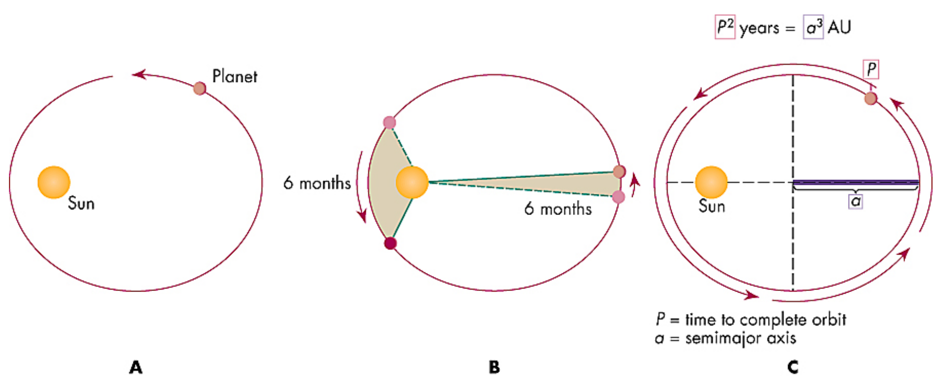
To maximize Δv:

- v_e should be as high as possible
- The mass of the rocket without its fuel (M_f) should be as small as possible
- Multistage rockets allow one to drop excess mass (that is not fuel) in order to keep the M_i/M_f ratio high.

Thrust = $M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right|$ Note that thrust increases with burn rate.

Kepler's Laws

- Tyco (1546 – 1601) Very meticulous and accurate observational data. Developed a hybrid of the geocentric and heliocentric models
- Kepler – (1571 – 1630) Tyco's student. Inherited his data. Determined from Tyco's data that the orbit of Mars was an ellipse, not a circle. Determined that the orbits of all planets must be ellipses. Developed *Kepler's Three Laws*



From *Explorations*, 5th Ed., by Tom Arny

- **K I** - Planets move in elliptical orbits with the Sun at one focus of the ellipse.
- **K II** - The orbital speed of a planet varies so that a line joining the Sun and the planet will sweep over equal areas in equal time intervals.
- **K III** - The amount of time a planet takes to orbit the Sun is related to its orbit's size. The square of the period (*P*) is equal to the cube of the semimajor axis (*a*):

$$P^2 = a^3$$

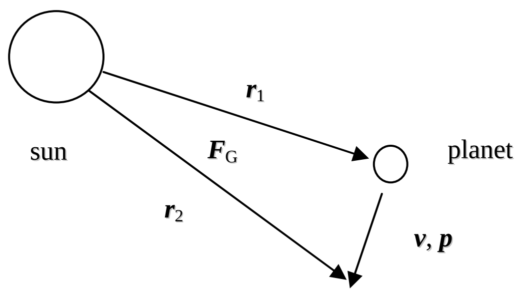
Elliptical orbits, Kepler II, and conservation of angular momentum.

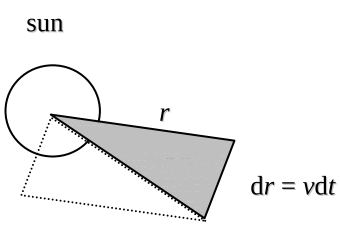
In an elliptical orbit the radius, *r*, changes during the course of the orbit. Note that ***r*** and ***F*** are parallel at each point along the planet's orbit, so:

$$\vec{\tau} = \vec{r} \times \vec{F} = r \times F(r)\hat{r} = 0 = \frac{d\vec{L}}{dt}$$

and

$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ is constant, since if $\tau = 0$, ***L*** is conserved





dA = ½ of the area of parallelogram of area *r* x d*r*

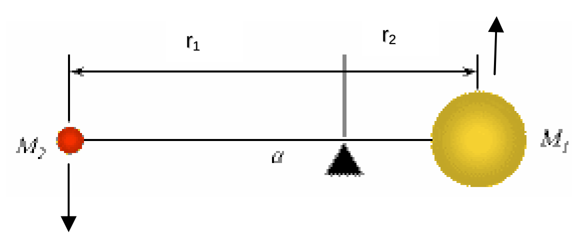
$$dA = \frac{1}{2}\left|r \times dr\right| = \frac{1}{2}\left|r \times vdt\right| = \frac{1}{2}\frac{mv}{m}dt = \frac{1}{2}\frac{L}{m}dt$$

$$\therefore \frac{dA}{dt} = \frac{L}{2m}$$

- The *sum* of all external torques is zero
- ***L*** has some constant value
- Conservation of ***L*** arises from the fact that gravity is a *central force*
- Angular momentum is constant for any object under the influence of a central force
- When planets are closer to the sun in elliptical orbits (perihelion) their orbital speed increases
- When planets are farther from the sun in elliptical orbits (aphelion) their orbital speed decreases
- **K II** does not depend upon the inverse square nature of the gravitational force

Newton's Version of Kepler III

- The external forces acting on the solar system are essentially zero so the momentum of the solar system is conserved.
- Consider two masses, m_1 and m_2, orbiting their stationary COM at the respective distances of r_1 and r_2.
- Because F_G acts only along the line connecting the centers of the two masses both must complete an orbit in the same period, *P*, though their speeds will be different.



$$F_{c1} = \frac{m_1 v_1^2}{r_1} = 4\pi^2 \frac{m_1 r_1}{P^2}$$

$$F_{c2} = \frac{m_2 v_2^2}{r_2} = 4\pi^2 \frac{m_2 r_2}{P^2}$$

By Newton III, F_{c1} = F_{c2}, hence: $\frac{r_1}{r_2} = \frac{m_2}{m_1}$ (the position of the COM)

Note: $a = r_1 + r_2$

So $r_1 = \frac{m_2 a}{m_1 + m_2}$, and since $F_1 = F_2 = F_G = G\frac{m_1 m_2}{a^2}$

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \qquad \text{Newton's form of Kepler III}$$

In our solar system $M_{sun} >> m_{planet}$ and $k = \frac{4\pi^2}{GM_{sun}}$ in Kepler III.

Solar System Calculations

Recall: $$P^2 = ka^3 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \qquad \text{Newton's form of Kepler's harmonic law}$$

In our solar system $M_{sun} >> m_{planet}$ and $k = \frac{4\pi^2}{GM_{sun}}$.

Now consider that in our solar system we measure sidereal periods (*P*) in years and semimajor axes (*a*) in AU the harmonic law for the sun and planets simplifies to:

$$\frac{a^3}{P^2} = 1 + \frac{m_p}{M_s}$$

This may be achieved by expressing the harmonic law as a series of ratio terms:

$$\left[\frac{(m_1 + m_2)}{(m_1^{'} + m_2^{'})}\right]\left(\frac{P}{P^{'}}\right)^2 = \left(\frac{a}{a^{'}}\right)^3$$

where the unprimed system is compared to the standard primed system.

- In our solar system (or in binary star systems) the standard system is Earth-Sun: *P* is in years, *a* is in AU, masses in solar masses ($M_{\odot}$)
- In these units *k* = 1 and *G* = $4\pi^2$ so $P^2 = a^3$.
- For planetary satellites the standard system is Earth-Moon: $P' = 27.3$ days, $a' = 3.84 \times 10^5$ km, and $\left(m_1' + m_2'\right)$ = $M_{\oplus}$ or 5.976 × 10^{24} kg; the calculation produces *P* in days, *a* in kilometers, and masses in Earth masses.

Example

A comet has orbital period of 7 years. What is its semimajor axis length, *a*?

Note: $M_{sun} + m \approx M_{sun}$, in this case the harmonic equation becomes (recalling that in solar system units *k* = 1 and *G* = $4\pi^2$ so $P^2 = a^3$):

$$P^{\frac{2}{3}} = a = 7^{\frac{2}{3}} = 3.7 \text{ AU}$$

Example

Miranda, a moon of Uranus, orbits the planet in about 1.4 days at an average distance of 128,000 km. What is the mass of Uranus?

Using the Earth-Moon standard system: $P' = 27.3$ days, $a' = 3.84 \times 10^5$ km:

$$\left[\frac{M_U + m_m}{M_{\oplus} + M_m}\right]\left(\frac{P_m}{P}\right)^2 \approx \left(\frac{M_U}{M_{\oplus}}\right)\left(\frac{P_m}{P}\right)^2 = \left(\frac{a_m}{a}\right)^3$$

$$M_U \approx \left(\frac{27.3}{1.4}\right)^2 \left(\frac{128{,}000}{384{,}000}\right)^3 M_{\oplus}$$

$$M_U \approx 14 M_{\oplus}$$

Example

Comet Encke has an orbital period of 3.3 years. What is its semimajor axis length?

$$P^{\frac{2}{3}} = a = 3.3^{\frac{2}{3}} = 2.2 \text{ AU}$$

Example

Europa orbits Jupiter once every 3.55 days at an average distance of 671,000 km. What is the mass of Jupiter?

$$\left[\frac{M_J + m_E}{M_\oplus + M_m}\right]\left(\frac{P_E}{P}\right)^2 \approx \left(\frac{M_J}{M_\oplus}\right)\left(\frac{P_E}{P}\right)^2 = \left(\frac{a_E}{a}\right)^3$$

$$M_J \approx \left(\frac{27.3}{3.55}\right)^2\left(\frac{671{,}000}{384{,}000}\right)^3 M_\oplus$$

$$M_J \approx 315 M_\oplus$$

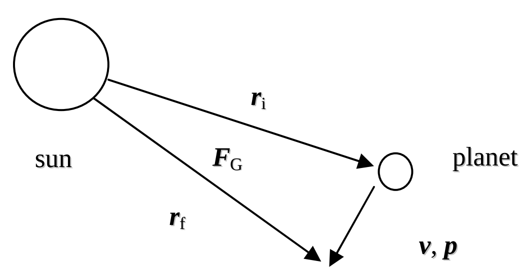
Energy Considerations in Planetary and Satellite Motion

Restrict ourselves to the case of

- One object (*m*) orbiting another (*M*) in a circular orbit
- *M* >>> *m*
- *M* is at rest in an inertial frame
- KE = $\frac{1}{2}mv^2$
- PE = $-G\frac{Mm}{r}$ (note: PE is negative for attractive forces)
- Energy is conserved if the system is isolated

Total Energy: ***E* = *K* + *U*** $\rightarrow$ ***E*** = $\frac{1}{2}mv^2 - G\frac{Mm}{r}$

$$E_i = E_f \therefore \frac{1}{2}mv_i^2 - G\frac{Mm}{r_i} = \frac{1}{2}mv_f^2 - G\frac{Mm}{r_f}$$



Gravity is a centripetal force:

$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

$$G\frac{Mm}{r} = mv^2$$

$$G\frac{Mm}{2r} = \frac{1}{2}mv^2$$

So:

$$E_{tot} \therefore G\frac{Mm}{2r} - G\frac{Mm}{r} = -G\frac{Mm}{2r}$$

- For a *bound* system ***E*** < 0
- Total energy is negative
- KE is positive and ½ the magnitude of PE
- ***E*** is the *binding energy* of the system
- Can be extended to elliptical orbits as well

Atmospheres of Planets and Satellites in our Solar System

- Mercury and Earth's moon have no atmosphere
- Venus and Mars have CO_2 atmospheres
- Earth has a N_2, O_2 atmosphere
- Jovian Planets have H_2, He atmospheres
- Ionospheres form when solar uv and x-rays ionize atoms or dissociate molecules high in a planetary atmosphere
- Ideal or perfect gas - all particles are point sources, only interactions between particles or particles and the container are elastic collisions. In such a gas:

$$PV = nRT \rightarrow PV = NkT \rightarrow P = nkT$$

where P = pressure (N/m^2), n is number density of particles (particles/m^3), T is the absolute temperature (K) and k is the Boltzman constant (1.38×10^{-23} J/K)

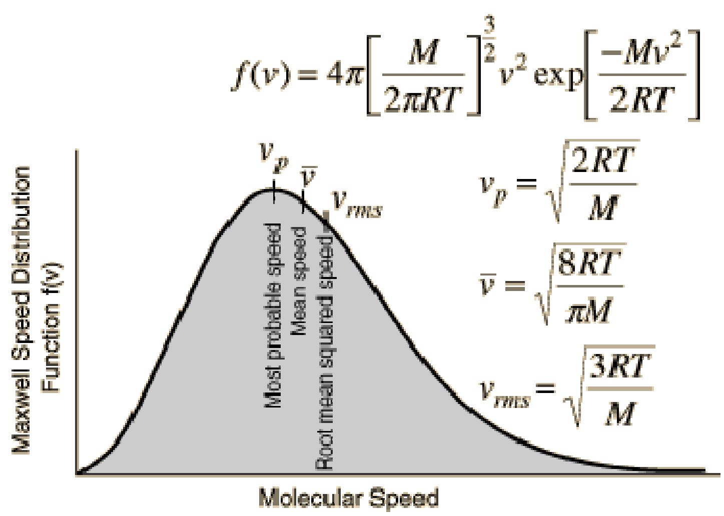
- In any collection of gas at a given temperature, T, collisions between particles occur in a equilibrium distribution of velocities:

$$f(v)dv \propto e^{-1/2mv^2/kT} v^2 dv$$

known as the Maxwell distribution of velocities.

- Note that this distribution has a long tail at high velocities (asymmetric) - a few particles are boosted to very high speeds by collisions.

- The peak of this distribution, indicated by v_p, is the most probable speed of a particle in this distribution.



http://hyperphysics.phy-astr.gsu.edu/hbase/kinetic/kintem.html

- It may be shown that:

$$v_p = \sqrt{\frac{2kT}{m}}$$

- The speeds of particles change violently after collisions but the average speed of all particles is v_p. The most probable speed increases with absolute temperature and decreases with mass.

- The average kinetic energy per particle is:

$$\langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$$

and increases with absolute temperature.

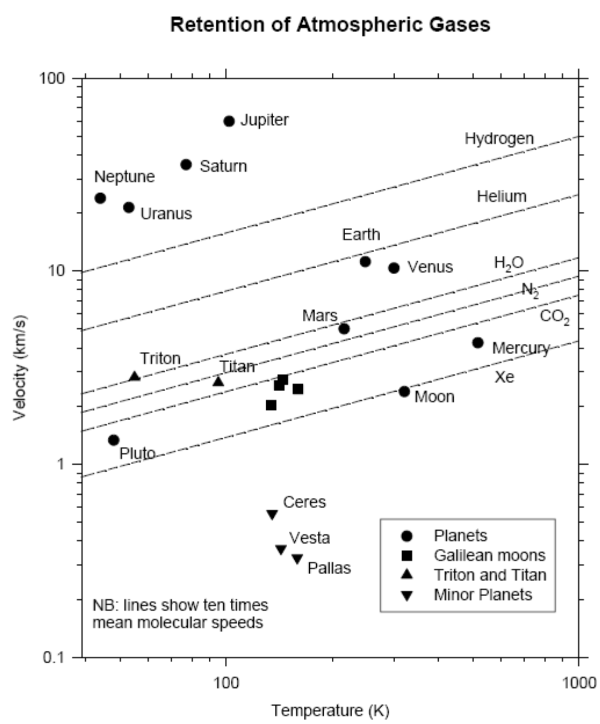
- The rms speed is:

$$v_{rms} = \langle v^2 \rangle^{1/2} = \sqrt{\frac{3kT}{m}}$$

and increases with absolute temperature and decreases with mass.

- Recall that escape velocity, $v_e = \sqrt{\frac{2GM}{r}}$, is the speed for which a particle is able to become "unbound" from a system.
- If v_{rms} = v_e a body will lose it's atmosphere in just a few days. For a planet or satellite to retain an atmosphere for 4 + billion years: $v_{rms} \leq 0.1\ v_e$.
- A given particle of gas is retained indefinitely by a planet or satellite when:

$$T \leq \frac{GMm}{150kr}$$



http://www.ita.uni-heidelberg.de/~wjd/SS 1

Example 1

The Earth orbits once around the Sun every 365 days. The average distance from Earth to the sun is 150 million km (1.5 $\times 10^{11}$ m). The Earth's mass is 6 $\times 10^{24}$ kg.

- How many seconds does it take the Earth to orbit the sun?
- What distance does the earth travel in one year (one orbit)?
- What is the Earth's average velocity in its orbit?
- Compute the Earth's acceleration.
- What is the source of the centripetal force?
- What is the other force in the force pair?
- What is the magnitude of the centripetal force acting on Earth (exerted by the sun)?

The number of seconds in a year is:

$$\frac{365days}{1year} \times \frac{24hours}{1day} \times \frac{60m}{1hour} \times \frac{60s}{1m} = 31{,}536{,}000s$$

The distance traveled in a year is:

Circumference of a circle = $2\pi r \Rightarrow (2)(3.14)(1.5 \times 10^{11} m) = 9.42 \times 10^{11} m$

The average velocity is:

$$\frac{9.42 \times 10^{11} m}{3.15 \times 10^{7} s} \approx 30{,}000 m/s$$

The acceleration is:

$a = \frac{v^2}{r} = \frac{(3 \times 10^4 m/s)^2}{1.5 \times 10^{11} m}$ or roughly 0.006 m/s^2! This is an exceedingly small value! Based on this do you think that the earth is an inertial reference frame at least with respect to orbital acceleration?

The source of the centripetal force is gravity $F_c = m\frac{v^2}{r} = G\frac{m_1 m_2}{r^2}$. The other force in the force pair is that the earth attracts the sun with an equal and opposite force. What is the effect on the sun?

The magnitude of the centripetal force is:

$F_c = m\frac{v^2}{r} = (6 \times 10^{24} kg)(6 \times 10^{-3} m/s^2) = 3.6 \times 10^{22} N = G\frac{m_1 m_2}{r^2}$. Could you use this information to compute the mass of the sun?

Example 2

The Earth spins once around it's own rotational axis about every 24 hours (diurnal motion). The average distance from Earth's surface to its center is 6.4 $\times 10^6$ m. The Earth's mass is 6 $\times 10^{24}$ kg.

- How many seconds does it take the Earth to spin once on its axis?
- What is the distance around the earth surface (circumference)?
- What is the average velocity at which objects on the Earth's surface move due to diurnal motion?
- Compute the acceleration of objects on the Earth's surface due to diurnal motion.
- Is there a centripetal force present? If so what is the source of the centripetal force?
- What is the other force in the force pair?
- What is the magnitude of the centripetal force acting on bodies on the earth's surface?

The number of seconds in a day is:

$$\frac{24hours}{1day} \times \frac{60m}{1hour} \times \frac{60s}{1m} = 86{,}400s$$

The distance around the earth's surface is:

Circumference = $2\pi r \Rightarrow (2)(3.14)(6.4 \times 10^6\, m) = 4.2 \times 10^7\, m$ (about 25,000 miles)

The average velocity is:

$\frac{4.2 \times 10^7\, m}{8.64 \times 10^4\, s} = 486 m/s$ (about 1100 mph)

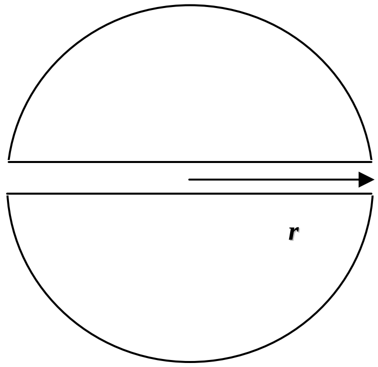
The acceleration is:

$a = \frac{v^2}{r} = \frac{(486m/s)^2}{6.4 \times 10^6\, m} \approx 3.7 \times 10^{-2}\, m/s^2 = 0.037 m/s^2$ This, again, is an exceedingly small value! Based on this do you think that the earth is an inertial reference frame at least with respect to spin acceleration?

The source of the centripetal force is gravity $F_c = F_g = mg = G\frac{m_1 m_2}{r^2}$. The acceleration due to gravity is 9.8 m/s^2. Note that this is much greater than the centripetal acceleration produced by the Earth's spin and therefore much greater than that required to keep objects on the surface of the earth from flying off into space due to the Earth's spin

Example 3

What would happen if one were to dig a hole all of the way through the earth and drop something into it.



It may be shown that for a particle inside of a sphere the force of gravity has the form:

$$F = -G\frac{mM}{R^3}r$$

it follows that: $a = \frac{F}{m} = -G\frac{mM}{R^3 m}r = -G\frac{M}{R^3}r$

It may be shown that the units of $G\frac{M}{R^3}$ are $\frac{\frac{N}{m}}{kg}$ which is the same as $\frac{k}{M} = \omega^2$

$\therefore a = -G\frac{M}{R^3}r = -\omega^2 r = -\omega^2 x$ and this system obeys Hooke's Law. Motion is simple harmonic!

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{GM}{R^3}}} = 5.06\times10^3 s = 84.3\,\text{min}$$

Questions for thought

- What would happen should the earth suddenly stop spinning on its axis? Would everything fly off into space?
- Why does earth not have much free hydrogen and little helium of any type?
- Our solar system is differentiated, i.e., the material in it is sorted largely by density. Based on our discussion of gravity what do you think accounts for this?
- We have considered the force of gravity between spherical objects. If one approachs a flat plate infinite in extent, the gravitational force depends only on its thickness, not on distance. Why?