

Fluidics

Hydrostatics

- Buoyancy (Archimedes Principle)
- Pascal's Principle
- Hydraulics

Hydrodynamics

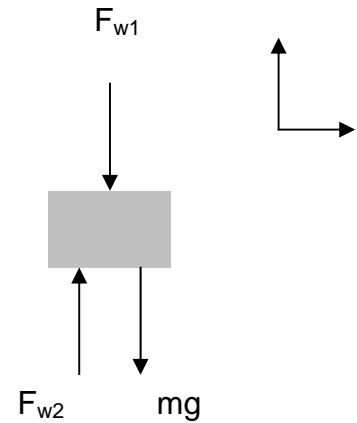
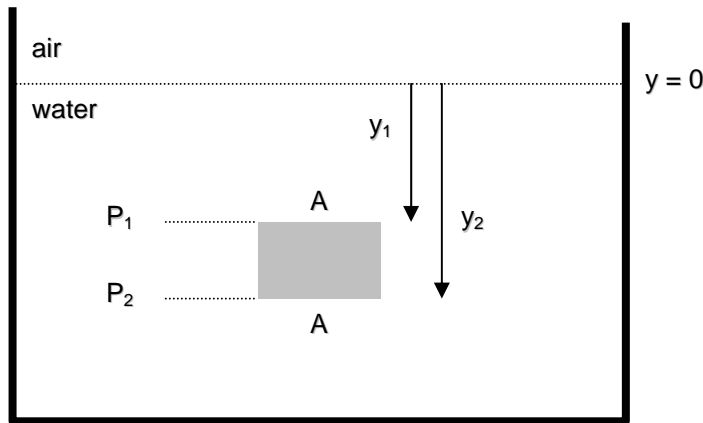
- Equation of Continuity
- Bernoulli's Equation

General Concepts

Fluids are substances that conform to the shape of any container in which they are placed. In general fluids are highly incompressible.

- Density - mass per unit volume: $\rho = \frac{m}{v}$. The SI unit of density is a kg/m^3 .
- Pressure - Force per unit area: $P = \frac{F}{A}$. Pressure is a vector that we simply define as a scalar for ease of use. The SI unit of pressure is a N/m^2 , also known as a Pascal.
- $1 \text{ atm} = 1.01 \times 10^5 \text{ Pa} = 760 \text{ torr (mm Hg)} = 14.7 \text{ lb/in}^2$.

Hydrostatics



$$\sum F_y = F_{w2} - F_{w1} - mg = 0$$

$$\therefore F_{w2} = F_{w1} + mg$$

$$P_2 A = P_1 A + mg$$

$$P_2 A = P_1 A + \rho V g$$

$$P_2 A = P_1 A + \rho g A (y_2 - y_1)$$

Note: If the top of the object is at the surface then P_1 is equal to atmospheric pressure. In this case: $y_1 = 0$, $P_1 = P_{atm}$, $y_2 = -h$, $P_2 = P$, then:

$$P = P_{atm} + \rho g h$$

- This result depends on depth (h) but not upon the surrounding volume of fluid. The pressure at the bottom of a fluid is the same for a very thin column as it is for an ocean.
- Absolute Pressure in any fluid contains two elements: P_a and $\rho g h$. The difference between these two quantities is known as *gauge pressure*.
- Gauge pressure is what pressure-sensing devices generally display.

Example 1 Why subs generally don't have windows.

A sub operates at a depth of 1000m. What force would be exerted by the seawater on an observation window, $r = 0.10$ meters in radius?

$$\rho_{\text{seawater}} = 1025 \text{kg} \cdot \text{m}^{-3}$$

$$P = P_{\text{atm}} + \rho gh$$

$$P_{1000\text{m}} = 1.013 \times 10^5 \text{N} \cdot \text{m}^{-2} + (1025 \text{kg} \cdot \text{m}^{-3})(9.8 \text{m} \cdot \text{s}^{-2})(1000\text{m})$$

$$P_{1000\text{m}} = 1.01 \times 10^7 \text{N} \cdot \text{m}^{-2}$$

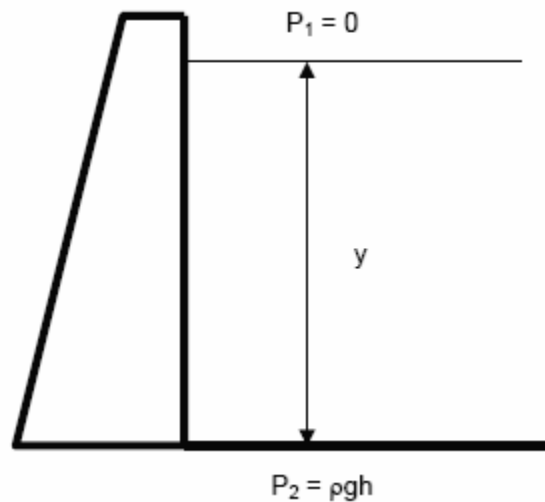
$$PA = F = (1.01 \times 10^7 \frac{\text{N}}{\text{m}^2})\pi(0.10\text{m})^2$$

$$F = 3.2 \times 10^5 \text{N}$$

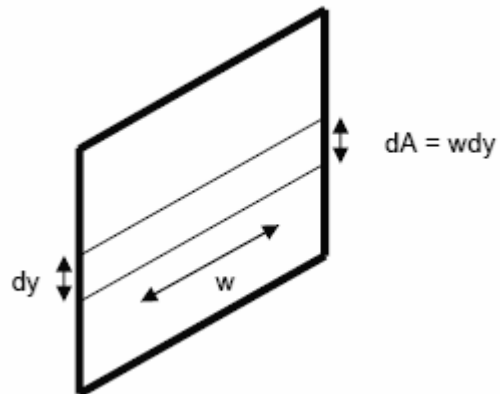
This is a large amount of force for a piece of glass or Plexiglas to sustain.

Example 2. Find the total force on a dam due to the water behind it.

Remember that the amount of water behind the dam makes no difference in the force on the dam (assuming a hydrostatic situation), only the depth. We'll also ignore the contribution of atmospheric pressure since we are interested in only the force of the water (why would the pressure of the air be of less interest?)



Since the force of the water varies with depth we must integrate over the height of the dam below waterline to obtain the desired total force. We do this by computing the force on a set of narrow strips of area dA and adding them up.



The force on any narrow strip of water is:

$$dF = PdA = \rho gyw dy$$

So the force on the entire dam is:

$$F = \int_0^h \rho gyw dy = \frac{1}{2} \rho g w h^2$$

Example 3. What is the total force and the absolute pressure on the bottom of a swimming pool 18.0 by 10.0 meters whose uniform depth is 2.0 meters? What will be the pressure against any side of the pool near the bottom?

- The size of the pool is irrelevant in determining the pressure in this problem, only the depth and the type of fluid (water).
- The size of the pool is necessary to determine the force.

The absolute pressure

$$P = P_{atm} + \rho gh$$

$$P_{2.0meters} = 1.013 \times 10^5 N \cdot m^{-2} + (1000 kg \cdot m^{-3})(9.8 m \cdot s^{-2})(2.0 m)$$

$$P = 1.2 \times 10^5 Pa$$

The total force

$$F = PA = (1.2 \times 10^5 Pa)(18.0 m \times 10.0 m) = 2.2 \times 10^7 N$$

The pressure against any side of the pool near the bottom:

$$P = 1.2 \times 10^5 Pa$$

Archimedes Principle

A body fully or partially immersed in a fluid is supported by a force equal to the weight of the fluid that the body displaces.

$$\text{Buoyant Force} = \text{Weight of the fluid displaced} = \rho_f V_f g$$

Example 4 A buoyant force of 26N acts on a piece of quartz that is completely immersed in ethyl alcohol. What is the volume of quartz?

The density of ethyl alcohol is 806 kg/m^3 . Using Archimedes Principle:

$$F_B = \rho_f V_f g \therefore \frac{F_B}{\rho_f g} = \frac{26 \text{ kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}}{(806 \text{ kg} \cdot \text{m}^{-3})(9.8 \text{ m} \cdot \text{s}^{-2})} = V_f = 3.3 \times 10^{-3} \text{ m}^3$$

Example 5 A solid, square wooden raft has an area of 16 m^2 and is 0.30 m thick. The density of the wood in the raft is 550 kg/m^3 . Does the raft float in water?

To start, we must determine the maximum buoyant force - that which would be achieved if the raft were completely submerged.

$$\rho = \frac{m}{V} \therefore \rho V = m \rightarrow (550 \text{ kg} \cdot \text{m}^{-3})(4.8 \text{ m}^3) = 2640 \text{ kg}$$

Weight of the raft:

$$\rho_{\text{wood}} V g \approx 26,000 \text{ N}$$

Weight of the water displaced by the submerged raft (buoyant force):

$$\rho_{\text{water}} V g \approx 47,000 \text{ N}$$

Since the buoyant force is greater than the weight of the raft, it floats!

Based on what you've learned about Archimedes Principle so far can you explain why boats made out of steel and even concrete can float?

Example 6. A geologist finds that a rock having a mass of 8.20kg in air has an apparent mass of 6.18kg when submerged in water. What is the density of the rock? $\rho_{\text{water}} = 1000\text{kg} \cdot \text{m}^{-3}$

Mass, of course, is a property of matter that does not depend on anything else besides how much matter is in an object.

For the given information we must determine the volume of the rock. To do this we note that the difference between the mass in the air and the apparent mass when measured underwater. The difference is due to the presence of a buoyant force.

$$\rho_{\text{water}} Vg = F_B = mg = (8.20\text{kg} - 6.18\text{kg})g \therefore V = 0.002\text{m}^3$$

Having determined the volume of the rock we can easily determine its density. We choose the mass of the rock in air because that is the "real" value.

$$\rho = \frac{m}{V} = \frac{8.20\text{kg}}{0.002\text{m}^3} = 4.1 \times 10^3 \text{kg} \cdot \text{m}^{-3}$$

Example 7. What fraction of a piece of aluminum will be submerged if it floats in Mercury?

To determine this, we proceed along the lines of Example 4, but now we must determine the portion of the aluminum that will be submerged.

$$\text{Mass of aluminum} = V_{al} \rho_{al}$$

$$\text{Mass of Mercury} = V_{Hg} \rho_{Hg}$$

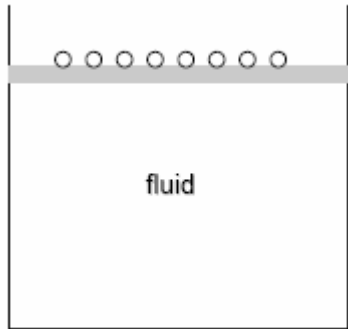
Since the aluminum floats it is in equilibrium and being supported by a buoyant force equal to the weight of the displaced mercury.

This leads to the conclusion:

$$V_{Al} \rho_{Al} = V_{Hg} \rho_{Hg} \therefore \frac{V_{Hg}}{V_{Al}} = \frac{\rho_{Al}}{\rho_{Hg}} = \frac{2.7 \times 10^3 \text{ kg} \cdot \text{m}^{-3}}{13.6 \times 10^3 \text{ kg} \cdot \text{m}^{-3}} = 0.199 \approx 20\%$$

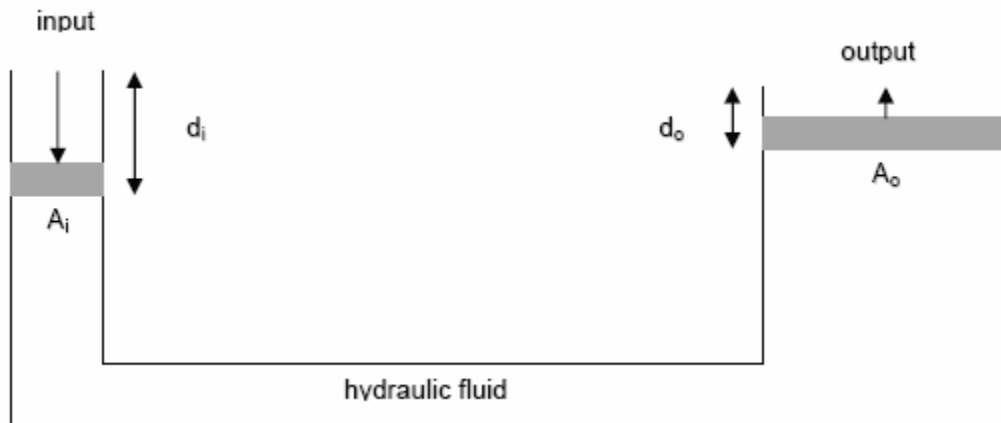
Pascal's Principle

A change in the pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of the containing vessel.



Adding more mass to the top of the cylinder increases the pressure on the top of the fluid which, in turn, is transmitted by the same increment to all parts of the fluid and to the walls of the cylinder in contact with the fluid.

Hydraulic Systems



According to Pascal's Principle, the pressure is the same throughout the fluid, hence:

$$P = \frac{F_i}{A_i} = \frac{F_o}{A_o} \therefore F_o = F_i \frac{A_o}{A_i} \text{ and by the same method } d_o = d_i \frac{A_i}{A_o}$$

Recalling that work = force x distance: $F_o d_o = (F_i \frac{A_o}{A_i})(d_i \frac{A_i}{A_o}) = F_i d_i$

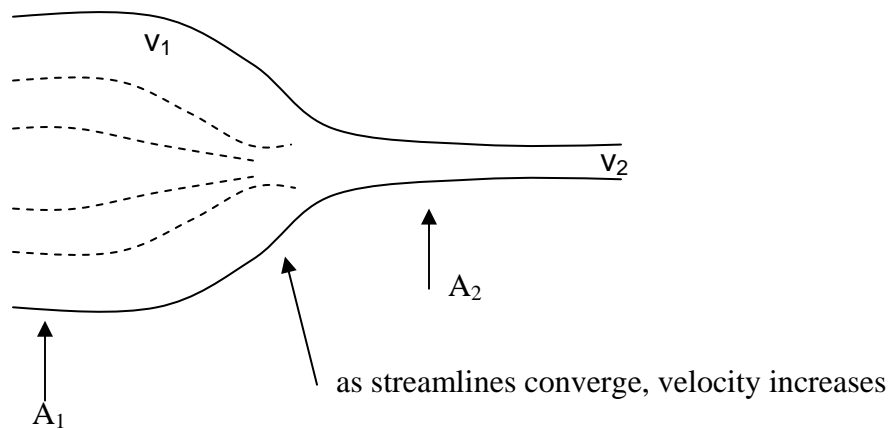
Work in = work out but a mechanical advantage is gained by trading force for distance.

Hydrodynamics

Ideal Fluids:

- Laminar Flow (Smooth)
- Incompressible Flow (Uniform density)
- Nonviscous Flow (no friction)
- Irrotational Flow (particles do not rotate about an axis through their center of mass)

Equation of Continuity



$$Av = \text{constant}$$

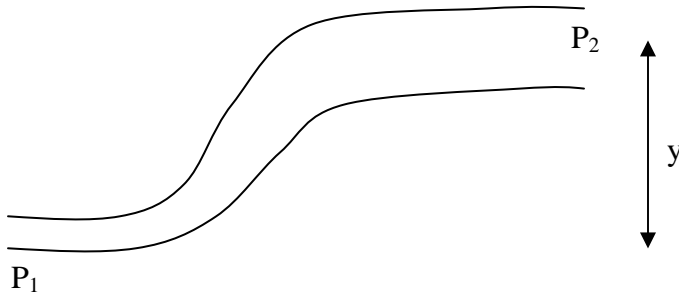
or

$$A_1v_1 = A_2v_2$$

Sometimes this is expressed in terms of mass flow rate or:

$$A_1v_1\rho = A_2v_2\rho \quad (\text{Mass flow rate})$$

Bernoulli's Equation



$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

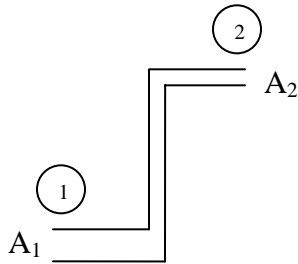
Bernoulli's equation takes into account changes in pressure due to elevation in the third and sixth terms, and changes in flow rate in the second and fourth terms.

Strategies:

- Identify points 1 and 2 in the system
- List: $P_1, P_2, v_1, v_2, y_1, y_2$

$\underbrace{\hspace{10em}}$	$\underbrace{\rho g}$
variables	constants
- In some problems, you will need to use the equation of continuity to relate $A_1 v_1 = A_2 v_2$
- The flow flux $\frac{dv}{dt}$ across any area A is: $\frac{dv}{dt} = Av$
- The corresponding mass flow flux is: $\frac{dm}{dt} = \rho Av$

Example 8 Water enters a house through a pipe 2.0 cm I.D. (1 cm radius) at an absolute pressure of 4×10^5 Pa. The pipe leading to the 2nd floor bathroom 5m above is 1.0 cm in diameter (0.5 cm radius). Velocity at the inlet is 4m/s. Find v and P at the bathroom.



$$v_1 = 4 \text{ m} \cdot \text{s}^{-1}$$

$$v_2 = ?$$

$$P_1 = 4 \times 10^5 \text{ Pa}$$

$$P_2 = ?$$

$$y_1 = 0$$

$$y_2 = 5\text{m}$$

If the water is running in the bathroom:

$$v_2 = \frac{A_1}{A_2} v_1 = \left(\frac{\pi(0.01\text{m})^2}{\pi(0.005\text{m})^2} \right) (4\text{m} \cdot \text{s}^{-1}) = 16\text{m} \cdot \text{s}^{-1}$$

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \rho g y_2 + \frac{1}{2} \rho v_2^2$$

$$P_2 = P_1 - \frac{1}{2} \rho (v_2^2 - v_1^2) - \rho g (y_2 - y_1)$$

$$P_2 = 2.3 \times 10^5 \text{ Pa} \approx 2.3 \text{ atm}$$

If the water is turned off in the bathroom:

$$P_2 = P_1 - \frac{1}{2} \rho (0) - \rho g (y_2 - y_1) \rightarrow P_2 = 3.5 \times 10^5 \text{ Pa}$$

Let's look again at Bernoulli's Equation: $P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$

In the second part of the previous example when flow is stopped the hydrodynamic situation becomes hydrostatic. All velocity terms in Bernoulli's equation go to zero and:

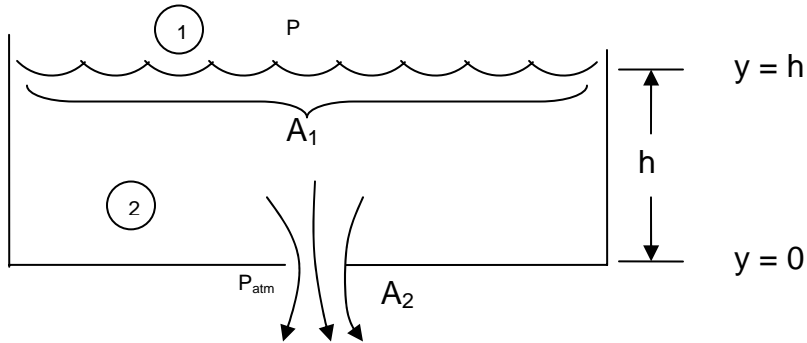
$$P_1 - P_2 = \rho g (y_2 - y_1)$$

and this reduces to:

$$P = P_{atm} + \rho g h \quad \text{as expected}$$

Other applications of Bernoulli's equation

Speed of efflux from a large container: **Toricelli's theorem**



Consider entire volume of fluid to be a flow tube:

$$P + \frac{1}{2} \rho v_1^2 + \rho g h = P_{atm} + \frac{1}{2} \rho v_2^2$$

$$v_2^2 = v_1^2 + 2 \frac{P - P_a}{\rho} + 2gh$$

if: $A_2 \ll A_1 \rightarrow v_1 \ll v_2$

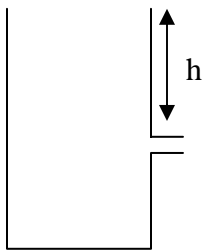
$$\Rightarrow v_2 = \sqrt{2 \frac{P - P_a}{\rho} + 2gh}$$

If the tank is open at the top then $P = P_{atm}$

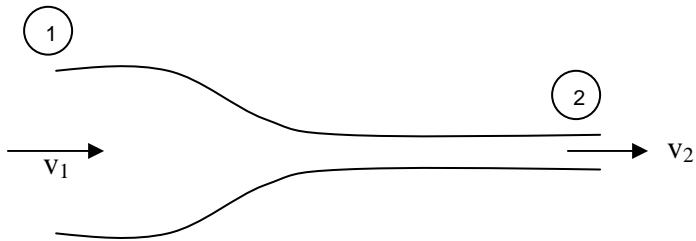
$$\Rightarrow v_2 = \sqrt{2gh}$$

Why is this result not surprising?

Toricelli's theorem also works for tanks where the outlet is not at the bottom:



Venturi Tubes



Since pipe is horizontal $y_1 = y_2$ and all terms that imply a difference in potential energy vanish (terms 3 and 6)

$$P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2$$

Recall: $v_1 = \frac{A_2}{A_1} v_2$

$$P_1 + \frac{1}{2} \rho \left(\frac{A_2}{A_1} \right)^2 v_2^2 = P_2 + \frac{1}{2} \rho v_2^2$$

$$v_2 = A_1 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

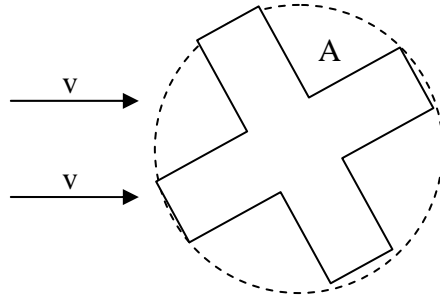
High flow velocity = low pressure - Low flow velocity = high pressure

$$A_2 < A_1 \Rightarrow P_1 > P_2$$

Energy from the Wind

Windmills convert kinetic energy of moving air to mechanical energy of a moving shaft. The average energy available per acre in the USA from wind is about 5KW. Assuming conversion is complete:

$$\frac{KE}{Volume_{air}} = \frac{1}{2} \rho v^2$$



The rate of flow through the blades is Av

$$Power = \frac{KE}{Volume_{air}} \times \frac{Volume_{air}}{Time} \rightarrow Power = \left(\frac{1}{2} \rho v^2 \right) (Av) = \frac{1}{2} \rho v^3 A$$

$$Intensity = \frac{Power}{Area} = \frac{1}{2} \rho v^3$$

Assume that the velocity of wind is on average about 12 m/s (≈ 30 mph). The density of the wind is about 1.3 kg/m³.

$$\frac{P}{A} = \left(\frac{1}{2} \right) (1.3 \text{ kg} \cdot \text{m}^3) (12 \text{ m} \cdot \text{s})^3 \approx 1100 \frac{\text{w}}{\text{m}^2} \text{ or } \approx 1.1 \frac{\text{kw}}{\text{m}^2}$$

- Contrast this with solar 1000 w/m² assuming ideal conditions and complete conversion.
- Note that power varies with cube of velocity, i.e., very sensitive to changes in windspeed (v).
- Conversion is, at best, approximately 60% complete: $\frac{P_{max}}{A} = \frac{8}{27} \rho v^3$ - and in real life, much less than this.

Example 9 A 15 cm radius air duct is used to replenish the air of a room 10 m × 5.0 m × 4.5 m every 10 minutes. How fast does air flow in the duct?

$$\text{Flow Rate} = \frac{\text{Volume}}{\text{Time}} = \frac{(10\text{m} \times 5\text{m} \times 4.5\text{m})}{600\text{s}} = 0.38\text{m}^3 \cdot \text{s}^{-1}$$

Note: $\text{m}^3/\text{s} = (\text{m}/\text{s})(\text{m}^2) = \text{velocity} \times \text{area} \therefore \frac{\text{m}^3/\text{s}}{\text{m}^2} = \text{m}/\text{s}$

$$v = \frac{0.375\text{m}^3 \cdot \text{s}^{-1}}{\pi(0.15\text{m})^2} = 5.3\text{m} \cdot \text{s}^{-1}$$

Example 10 Water at a pressure of 3.8 atm flows into the street level of an office building at a speed of 0.60 m/s through a pipe 5.0 cm in radius. The pipe tapers to 2.6 cm in radius by the top floor 20 m above. Compute the flow velocity and pressure at the top floor. All pressures are gauge pressures.

$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2$$

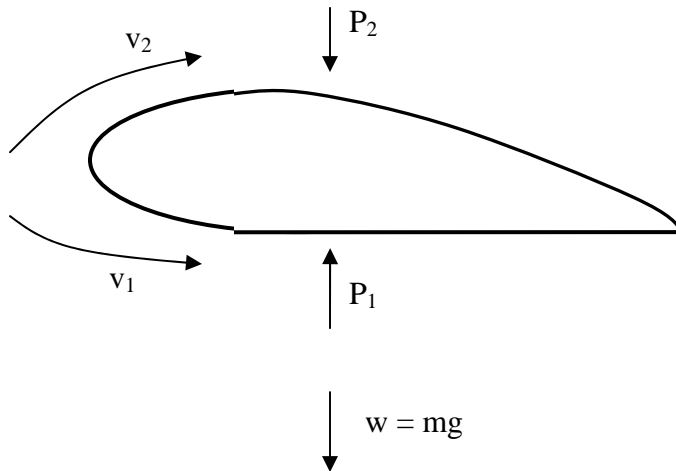
To find v_2 use the equation of continuity: $A_1 v_1 = A_2 v_2$

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{(\pi)(.05m)^2 (0.60m \cdot s^{-1})}{(\pi)(0.026m)^2} \rightarrow v_2 = 2.2m \cdot s^{-1}$$

$$P_2 = 3.8 \times 10^5 Pa + \frac{1}{2} (1000kg \cdot m^3) (0.6m \cdot s^{-1})^2 - \frac{1}{2} (1000kg \cdot m^3) (2.2m \cdot s^{-1})^2 - (1000kg \cdot m^3) (9.8m \cdot s^{-2}) 20m$$

$$P_2 = 3.8 \times 10^5 Pa + 180Pa - 2420Pa - 2.0 \times 10^5 Pa \rightarrow P_2 = 1.8 \times 10^5 Pa$$

Example 12 Each wing of an airplane has an area of 25 m^2 . The speed of the air is 50 m/s over the lower wing surface and 65 m/s over the upper wing surface. Assume that the airplane is in straight and level flight at about 300 meters above sea level and that all of the lift is supplied by the wings. What is the weight of the airplane?



Equilibrium in the y direction requires $F_{lift} = F_{weight}$ for straight and level flight.

$$(P_1 - P_2)A = mg$$

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \quad (\text{Horizontal flow})$$

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

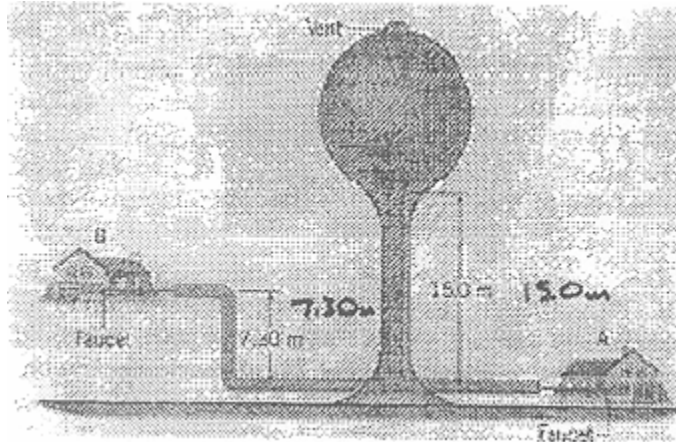
$$\frac{1}{2}\rho(v_2^2 - v_1^2)A = mg$$

$$mg = \frac{1}{2}(1.29 \text{ kg} \cdot \text{m}^{-3}) \left((65 \text{ m} \cdot \text{s}^{-1})^2 - (50 \text{ m} \cdot \text{s}^{-1})^2 \right) 50 \text{ m}^2 \quad (\text{2 wings = 50 square meters total) surface)}$$

$$mg = 5.56 \times 10^4 \text{ N}$$

So the weight of the plane is $5.56 \times 10^4 \text{ N}$

Example 13 The purpose of a water tower is to provide storage capacity and to provide sufficient pressure in the pipes that deliver the water to customers. The drawing below shows a spherical reservoir that contains 5.25×10^5 kg of water when full. The tower is vented to the atmosphere at the top. For a full reservoir find the gauge pressures in each house.



We need the total height, h , to the top of the water tower in both cases. Since we know the mass and the density of water we can compute its volume and therefore the diameter of the tank. This distance must be added on to the difference in height from the bottom of the tank to the ground to get the total height of the column of water.

In general: $P = P_a + \rho gh$

Gauge Pressure: $P_g = \rho gh$

$$M_{\text{water}} = \rho v_{\text{water}} = \rho \frac{4}{3} \pi r^3$$

Given 5.25×10^5 kg of water in the tank with a density of 1000 kg/m^3 , the radius of the tank is:

$$r = \left(\frac{3M}{4\pi\rho} \right)^{1/3} \rightarrow r = 5.0\text{m} \rightarrow d = 10.0\text{m}$$

For house A: $P_g = \rho gh = (1000\text{kg} \cdot \text{m}^{-3})(9.8\text{m} \cdot \text{s}^{-2})(25\text{m})$

$$P_g = 2.45 \times 10^5 \text{ Pa}$$

For house B: $P_g = \rho gh = (1000\text{kg} \cdot \text{m}^{-3})(9.8\text{m} \cdot \text{s}^{-2})(17.7\text{m})$

$$P_g = 1.73 \times 10^5 \text{ Pa}$$