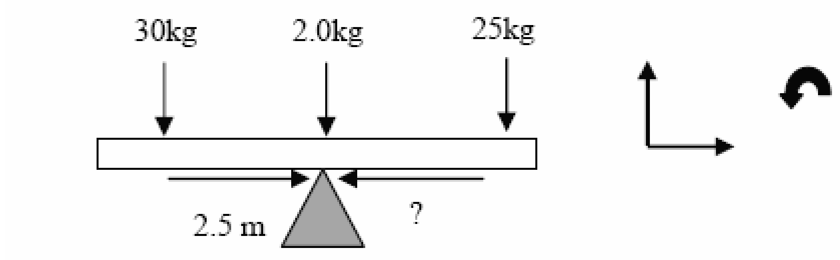
The First and Second Conditions for Equilibrium

The first condition for equilibrium: $\Sigma F = 0$
The second condition for equilibrium: $\Sigma \Gamma = 0$

- In when both of these conditions are satisfied in static systems all forces and torques sum to zero.
- In problems where the first and second conditions of equilibrium are satisfied, the best strategy is to create FBD's for both the first and second conditions, derive equations based on these FBD's and then see what useful information may be gleaned from these equations.
- When applying the second condition we are free to choose *any* axis about which to compute torques. It is best to choose an axis that eliminates one or more forces that have lines of force that pass through it.

Example 1 Consider a playground seesaw. The mass of the plank is 2.0 kg, the masses of two children on it are 25 kg and 30 kg with the 30 kg child sitting 2.5 meters from the center of the plank (the fulcrum) as shown below. Where must the second child sit in order for this system to be in equilibrium?



Noting that a normal force directed upwards acts at the point of the fulcrum, the FBD's for the **first condition** yield:

$$\sum F_y = N - m_{c1}g - m_p g - m_{c2}g = 0 \ \rightarrow \ \sum F_y = N - 294N - 19.6N - 245N = 0$$

Note that while this is all true it is not, by itself, particularly useful.

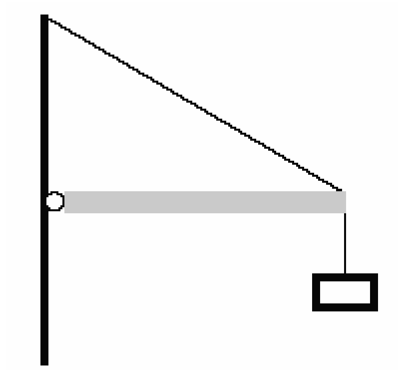
To apply the **second condition** we must first choose an axis about which to compute torques. The axis that makes the most physical sense would be one directly through the board over the fulcrum, but we could choose any axis that made computations easier. In this case choosing the axis associated with the fulcrum eliminates the forces created by the mass of the board itself since these act on the center of mass of the board which is located directly over the fulcrum.

The FBD's for the **second condition** yield:

$$\sum \Gamma = (294N)(2.5 meters) - (245N)(x meters) = 0$$

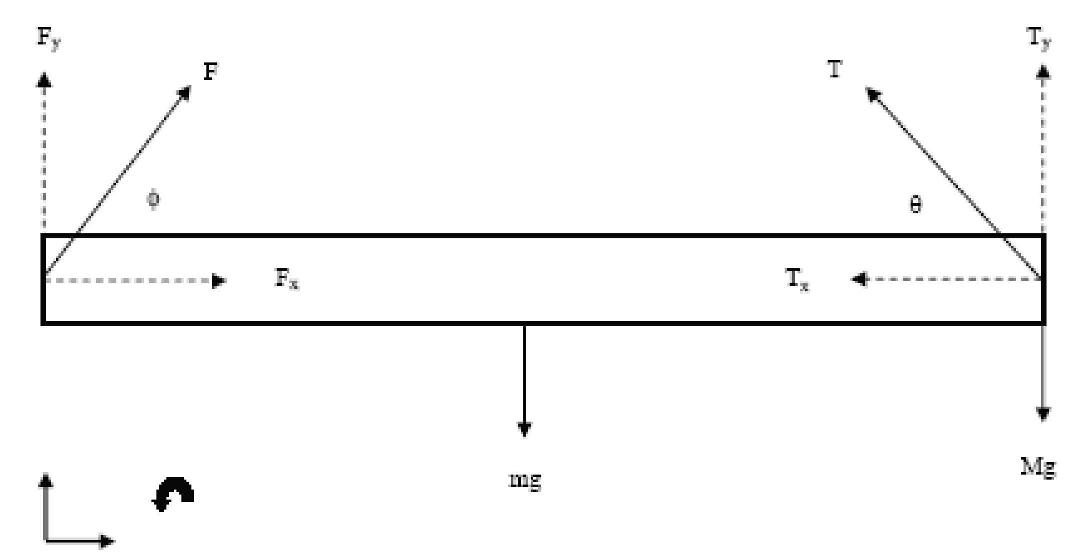
Solving this equation for *x* yields a distance of 3 meters.

Example 2 Consider the following cantilevered beam:



The beam has a mass of *m* = 25 kg and is 2.2 meters long. The suspended block has a mass *M* = 280 kg and the supporting cable makes an angle of 30^0 with the beam.

Determine the force that the wall exerts on the beam at the hinge and determine the tension in the supporting cable.



- Notice that the normal force is the *x* component of the force exerted by the wall on the beam through the hinge (F_x).
- Because the beam is also held up by the hinge (F_y) the total force the wall exerts on the beam is the aggregate of these two components. So we must determine, from the available information, *Fx*, *Fy*, *Tx*, *Ty* and finally *T* and *F*.

Application of the **first condition** with our sign convention yields:

$$\sum F_x = F_x - T_x = 0 \therefore F_x = T_x$$

$$\sum F_y = F_y + T_y - mg - Mg = 0$$

Application of the **second condition** with respect to the hinge yields:

F_y – line of action passes through the hinge $\rightarrow$ no torque
F_x – line of action passes through the hinge $\rightarrow$ no torque
T_x – line of action passes through the hinge $\rightarrow$ no torque

mg – exerts a torque
Mg – exerts a torque
T_y – exerts a torque

With our sign convention:

$$\sum \Gamma = -(mg)(1.1m) - (Mg)(2.2m) + (T_y)(2.2m) = 0$$

$$\sum \Gamma = -(245N)(1.1m) - (2744N)(2.2m) + (T_y)(2.2m) = 0$$

$$\sum \Gamma = -270N \cdot m - 6037N \cdot m + (T_y)(2.2m) = 0 \rightarrow \ T_y = 2867N$$

Since $T_y = T \sin\theta$: $\vec{T} = 5734N$, and with a little more work $T_x = 4966N$.

With the magnitude of T and all of its components known, it is a simple matter to substitute into the equation in y from the first condition and solve for F_y:

$$\sum F_y = F_y + 2867N - 245N - 2744N = 0$$

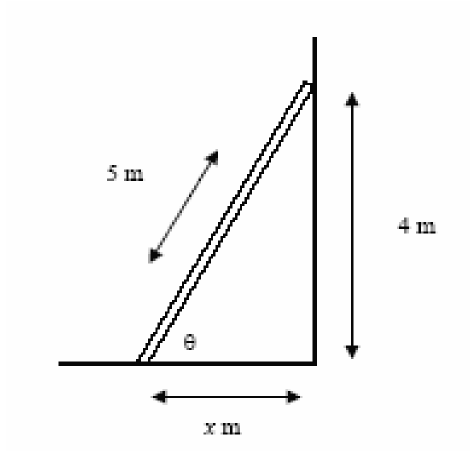
$$F_y = 122N$$

Noting that $F_x = T_x$ (why?), F_x = 4966N $\rightarrow$ $F = \sqrt{(4966N)^2 + (122N)^2} = 4967N$

$$\tan\phi = \frac{y}{x} = \frac{122N}{4966N} \therefore \phi = 1.4°$$

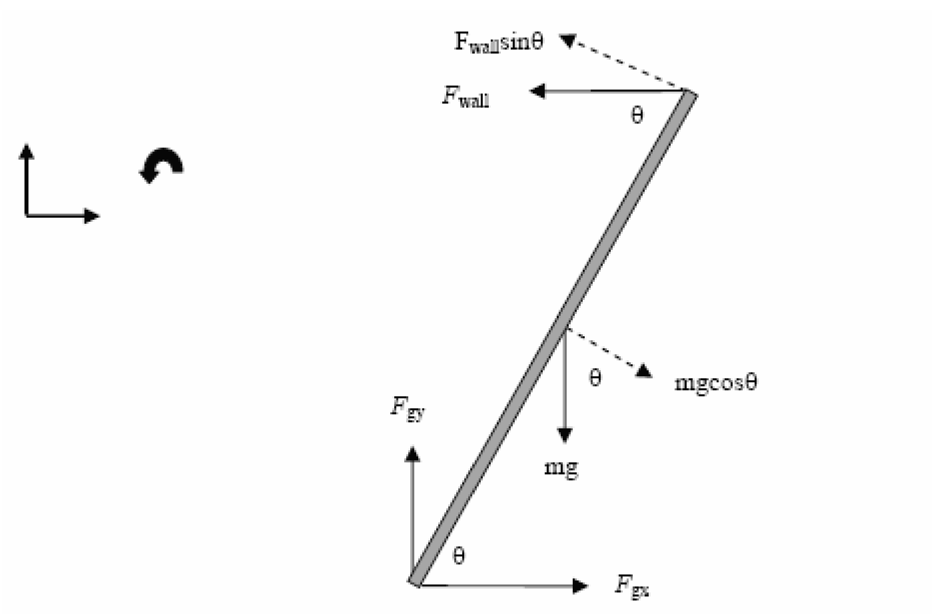
$$\vec{F} = 4967\text{ N @ } 1.4°$$

Example 3. A 5 meter long ladder leans against a frictionless wall. The point of contact between the ladder and the wall is 4 meters above the ground. The ladder is uniform with a mass of 12 kg. Determine the forces exerted by the ground and wall on the ladder.



A little yields $\theta = 53°$ and $x = 3$m.

FBD



Notice that since the wall is frictionless the force that it exerts on the latter is normal to the surface of the wall. It is necessary to find the component perpendicular to the ladder only for the purpose of computing a torque. The force that the ground exerts on the ladder, however, does have two components. Can you explain why?

Applying the **first condition** yields:

$$\sum F_y = F_{gy} - mg = 0$$

$$F_{gy} = mg \therefore F_{gy} = 118N$$

$$\sum F_x = F_{gx} - F_{wall} = 0$$

$$F_{gx} = F_{wall}$$

Applying the second condition with respect to the point of contact between the ground and latter (this eliminates F_g and it's components from torque computations):

$$\sum \Gamma = (F_{wall})(\sin\theta)(5m) - (mg)(\cos\theta)(2.5m) = 0$$

$$\sum \Gamma = (F_{wall})(4m) - (71N)(2.5m)$$

$$F_{wall} = 44N$$

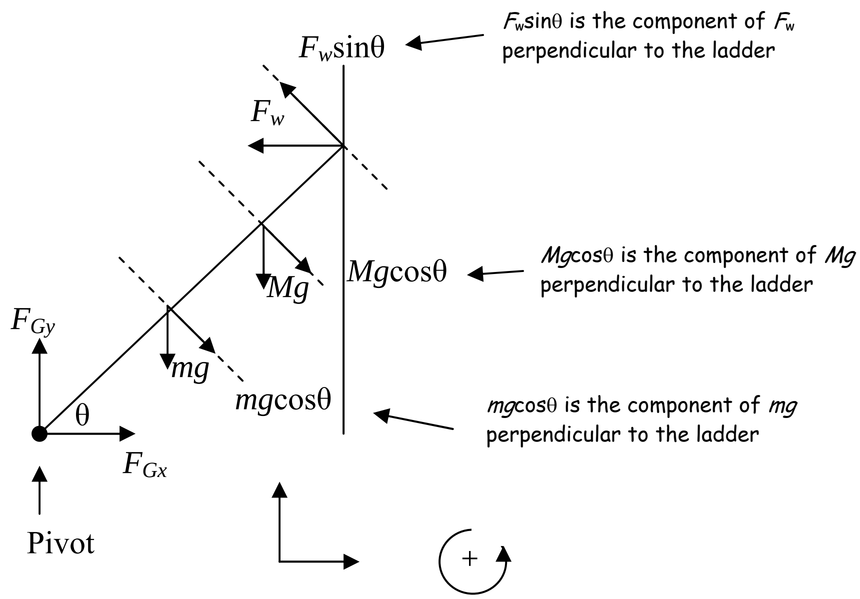
Recall that $F_{gx} = F_{wall} \therefore F_{gx} = 44N$ and:

$$F_g = \sqrt{(44N)^2 + (118N)^2} = 126N$$

$$\tan\phi = \frac{y}{x} = \frac{118N}{44N} \therefore \theta = 70°$$

$$\bar{F}_g = 126N \text{ @ } 70°$$

Now lets use the same ladder but with a 60kg painter 3.5 meters up the ladder and determine what the coefficient of friction, μ, is between the ladder and the floor.



$$\sum F_x = F_{Gx} - F_w = 0 \quad \therefore F_{Gx} = F_w$$

$$\sum F_y = F_{Gy} - mg - Mg = 0$$ Note: $F_{Gx} = \mu F_{Gy}$

$$F_{Gy} = (m+M)g \;= (12kg + 60kg)\left(9.8m \cdot s^{-2}\right)$$

$F_{Gy} = 706N$ (How does this compare to the earlier result?)

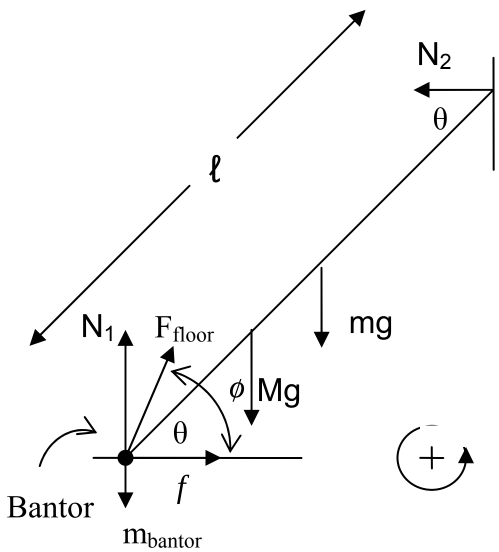
Recall: $\theta = 53°$

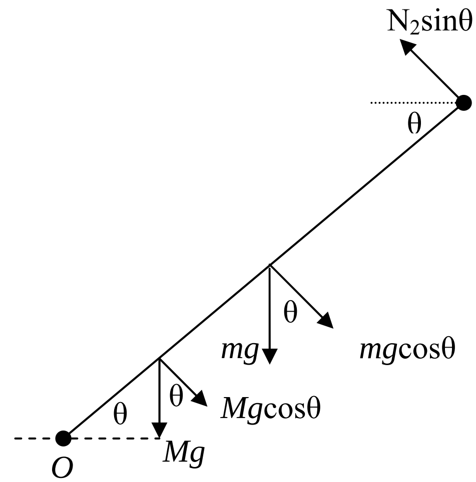
$$\sum \Gamma = (F_w)(\sin\theta)(5m) - (Mg)(\cos\theta)(3.5m) - (mg)(\cos\theta)(2.5m) = 0$$

$F_w = \dfrac{1239N \cdot m + 178N \cdot m}{4m} = 354N$ (How does this compare to the earlier result?)

$$F_w = \mu F_{Gy} \quad \therefore \mu = \frac{F_w}{F_{Gy}} = .5$$

Example 4. *A system consists of a ladder length ℓ and mass m on a rough floor leaning against a smooth vertical wall. A man of mass M is standing 1/3 of the way up the ladder from the floor. His dog, Bantor, stands on the ladder at its foot. The angle between the ladder and the floor is θ. Draw a free body diagram for this system and express the first and second conditions of equilibrium in terms of ℓ and θ about an axis at the base of the ladder.*





1st Condition:

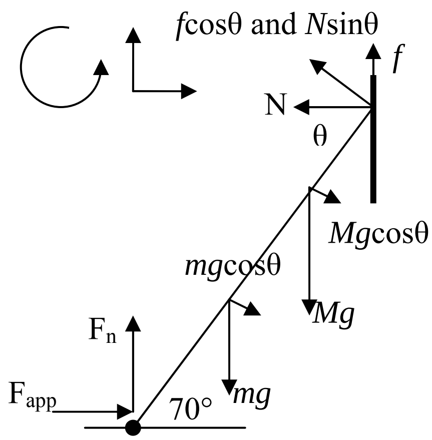
$$\sum F_x = f - N_2 = 0$$
$$\sum F_y = N_1 - Mg - mg - m_{bantor}g = 0$$

Bantor's mass exerts no torque since he stands at the origin

2nd Condition:

$$\sum \Gamma = N_2 \sin\theta(\ell) - mg\cos\theta\left(\frac{\ell}{2}\right) - Mg\cos\theta\left(\frac{\ell}{3}\right) = 0$$

Example 5. The maintenance staff in a hockey arena wishes to hang a scoreboard on the wall. They use a ladder with a uniform mass of 25kg to do the job. The ladder is 10 meters in length and an 80kg worker needs to be 7/10 of the way up the ladder to hang the scoreboard. The base of the ladder rests on a very smooth (icy) surface. If the coefficient of friction between the top of the ladder and the wall is 0.7, and the angle the ladder makes with the ice is 70°, what force must an assistant supply (who is presumably standing on a non-slippery surface) to the base of the ladder so that it does not slip?



1st condition:

$$\sum F_x = F_{app} - N = 0 \therefore F_{app} = N$$

$$\sum F_y = F_n - mg - Mg + f = 0 \because F_n + \mu N - mg - Mg = 0$$

2nd condition:

$$\sum \Gamma = (-mg\cos\theta)(5m) - (Mg\cos\theta)(7m) + (N\sin\theta)(10m) + (f\cos\theta)(10m) = 0$$

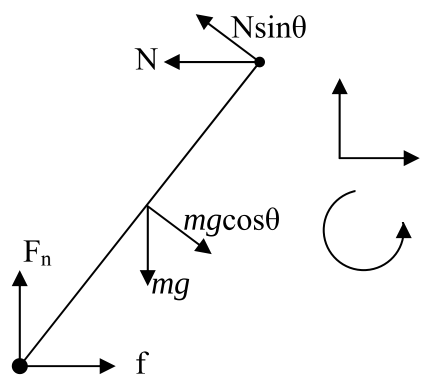
$$\big((25kg)(9.8m\cdot s^{-2})(\cos 70°)(5m)\big) + \big((80kg)(9.8m\cdot s^{-2})(\cos 70°)(7m)\big) =$$
$$\big((N)(\sin 70°)(10m)\big) + \big(\mu N(\cos 70°)(10m)\big)$$

Note: $N = F_{app}$ (from the 1st condition)

$$419N\cdot m + 1877N\cdot m = F_{app}(9.39 + 2.4)m$$

$$F_{app} = 195N$$

Example 6. A uniform ladder is leaning against a smooth, vertical wall. The ladder makes an angle of θ with the ground. If the coefficient of friction between the ladder and the ground is 0.70, find the minimum value of θ such that the ladder does not slip.



1st condition:

$$\sum F_x = f - N = 0 \quad \because \quad f = N = \mu F_n = \mu mg$$
$$\sum F_y = F_n - mg = 0 \quad \therefore \quad F_n = mg$$

2nd condition:

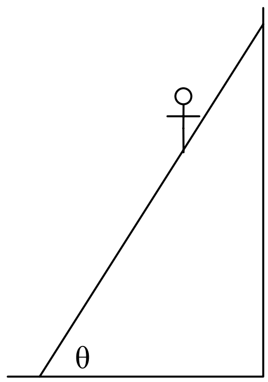
$$\sum \Gamma = -mg\cos\theta\left(\frac{\ell}{2}\right) + N\sin\theta(\ell) = 0 \because mg\cos\theta\left(\frac{\ell}{2}\right) = \mu mg\sin\theta(\ell)$$

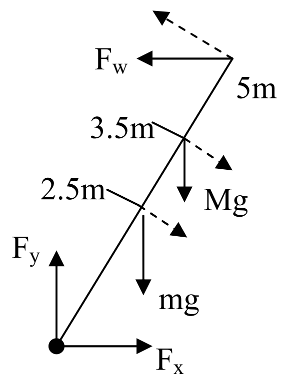
$$\cos\theta\left(\frac{1}{2}\right) = \mu\sin\theta$$

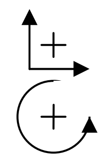
$$\frac{1}{2\mu} = \tan\theta$$

$$\theta = 35.5°$$

Example 7. A ladder (m = 1.20kg) leans against a frictionless wall 4.0 meters above the ground. The base of the ladder is 3.0 meters from the wall. A 60kg painter stands 70% of the way up the ladder. Assuming that the ladder is stable what is the minimum coefficient of static friction between the ladder and the ground?







Use the Pythagorean Theorem to solve for the length of the ladder:

$$a^2 + b^2 = c^2 \quad \therefore \quad c = 5m$$

Noting that this is a $3-4-5 \quad triangle \quad \theta \approx 53°$

1st Condition:

$$\sum F_x = F_x - F_w = 0 \quad \therefore \quad F_x = F_w$$

$$\sum F_y = F_y - mg - Mg = 0 \quad \therefore \quad F_y = (m+M)g = (1.20kg + 60kg)\left(9.8m \cdot s^{-2}\right) = 600N$$

Note: $F_x = f_s = \mu F_y \therefore \mu(600N) = F_w$

2nd Condition:

$$\sum \Gamma = -mg\cos\theta(2.5m) - Mg\cos\theta(3.5m) + F_w \sin\theta(5m) = 0$$

$$mg\cos\theta(2.5m) + Mg\cos\theta(3.5m) = \mu(600N)\sin\theta(5m)$$

$$\mu = \frac{(1.20kg)(9.8m \cdot s^{-2})\cos(53°)(2.5m) + (60kg)(9.8m \cdot s^{-2})\cos(53°)(3.5m)}{(600N)\sin(53°)(5m)} = 0.52$$