Experimental Statistics Part II

Combining Errors

Suppose that you were given the assignment of accurately determining the area of a tabletop. You have at your disposal a meter stick and the table. You are expected to produce a result that is as accurate as possible with the instruments given and to state the uncertainty in the measurement.

You would probably proceed by taking a number of measurements of each critical dimension, i.e., length and width, with the meter stick. Because you would know the accuracy available with the meter stick you would be able to state the uncertainty in each measurement, e.g., $L \pm \sigma$, and $W \pm \sigma$. You would then combine these measurements in an equation that yields the area of the tabletop. But how would you carry the uncertainties in measurement through your calculations in such a manner as to yield the correct uncertainty in the area?

After making five measurements of the length and width you find them to be:

$$L = 200 \pm 2cm \;\&\; W = 110 \pm 1cm$$

Given these measurements, what is the area of the tabletop and the error in the result? The rules for combining errors may be stated:

1. If $f = x + y$ then $\Delta f = \Delta x + \Delta y$
2. If $f = x - y$ then $\Delta f = \Delta x + \Delta y$
3. If $f = x \times y$ then $\frac{\Delta f}{f} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$
4. If $f = x \div y$ then $\frac{\Delta f}{f} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$
5. If $f = x^n$ then $\frac{\Delta f}{f} = n\frac{\Delta x}{x}$

Where Δ represents the error in the measurement. Notice that in the second instance the minus sign is suppressed. This means that the expected error in f is always a conservative (pessimistic) estimate. In other words, the assumption is that errors always combine in the worst possible way.

These five equations may easily be developed with a little calculus. Consider, for example, the function $f = x \times y$. The total differential of f is: $df = xdy + ydx$ or $\frac{df}{f} = \frac{dy}{y} + \frac{dx}{x}$, where the quantities d$f$, d$x$, and d$y$ are the errors in f, y, and x respectively.

The mean area of the tabletop with dimensions $L = 200 \pm 2cm$ & $W = 110 \pm 1cm$ is $2.20 \times 10^4 \, cm^2$, and the error is:

$$\Delta A = A\left(\frac{\Delta L}{L} + \frac{\Delta W}{W}\right) = 2.20\times10^4\left(\frac{2}{200} + \frac{1}{110}\right) = 0.04\times10^4\,cm^2$$

So the area with the uncertainty in measurement is stated:

$$A = 2.20\times10^4\,cm^2 \pm 0.04\times10^4\,cm^2 \text{ or } A = (2.2 \pm 0.04)\times10^4\,cm^2$$

Now suppose that you were given the assignment of accurately determining the density of a cylinder of unknown composition. You have at your disposal the cylinder, a mass balance, and a set of vernier calipers. You are, again, expected to produce a result that is as accurate as possible with the instruments given and to state the uncertainty in the measurement.

You would again proceed by taking a number of measurements of each critical dimension, i.e., length and radius, with the calipers. Because you would know the accuracy available with the vernier calipers you would be able to state the uncertainty in each measurement, e.g., $r \pm \sigma$ (Δr), and $h \pm \sigma$ (Δh). You would then combine these measurements in an equation that yields the volume of the cylinder, and ultimately, the density.

$$\Delta V = V\left(\frac{2\Delta r}{r} + \frac{\Delta h}{h}\right) \text{ and } \Delta d = d\left(\frac{\Delta m}{m} + \frac{\Delta v}{v}\right)$$