Potential Difference, Electrical Potential Energy and Electrical Potential

- Gauss's Law relates electric flux to electric charge.
- Coulomb's Law relates electric force to electric charge.
- We seek a method of relating $\boldsymbol{F}_E$ and $\boldsymbol{E}$ to the work done moving a particle around in an $\boldsymbol{E}$ field.
- Recall that $\boldsymbol{F}_E$ is a conservative force
- Recall that in general $U = -\int \vec{F}(r) \cdot d\vec{r}$

When a test charge is moved from A to B in an electric field, the work done on the charge is:

$$\int \vec{F} \cdot d\vec{s} = q' \vec{E} \cdot \vec{s}$$

- A charged particle that is free to move in an electric field is analogous to a massive object that is free to move in a gravitational field.
- The exception is that massive objects move in only one direction (left to their own devices) in a gravitational field
- Just as an external force performs work in displacing an object uphill and the gravitational force does work in displacing an object downhill in a gravitational field, an external force does work in displacing a charge from a region of lesser electric potential to a region of greater electrical potential and vice versa in an electrical field.

We previously defined the work done by a conservative force in displacing an object as the negative of the change in potential energy, ΔU. In the case of the electric force:

$$dU = -q' \vec{E} \cdot d\vec{s}$$

For the displacement A to B the change in potential energy is:

$$\Delta U = U_B - U_A = -q' \int_A^B \vec{E} \cdot d\vec{s}$$

- This integral from A to B is known as a *path integral* or a *line integral.*
- Since the force involved (the Coulomb force) is conservative, the work does not depend upon the path taken between A and B.

Potential Difference

The potential difference between two points in an electric field, A and B, denoted $V_B - V_A$ is defined as the change in potential energy divided by a charge defined as a test charge, q':

$$V_B - V_A = \frac{U_B - U_A}{q'} = -\int_A^B \vec{E} \cdot d\vec{s}$$

- Potential difference is not the same as potential energy!
- PD (ΔV) is proportional to PE (ΔU). The two are related by a factor of q'.

$$\Delta V = \frac{\Delta U}{q'} = -\int_A^B \vec{E} \cdot \vec{s}$$

- Because PE is scalar, PD is also a scalar.
- PD may be thought of as the potential energy per unit charge.
- The change in potential energy of a charge is the negative of the work done by the electric force.
- The potential difference $V_B - V_A$ is equal to the work per unit charge that an external agent must perform to move a test charge from A to B without a change in kinetic energy.
- Notice that we have defined only *changes* in potential. That is because only differences in V are meaningful.
- The electric potential function is often taken to be zero at some convenient reference point. Usually (though not always) this reference point is chosen to be a point remote enough from the charge or charge distribution producing the electric field that it may be taken as infinity. If this is the case, the electric potential at any point P is:

$$V_p = -\int_\infty^P \vec{E} \cdot d\vec{s}$$

  where V_P is the work per unit charge required to bring a charge from infinity to point P.
- The unit of electrical potential is the *Volt*. $1\text{V} \equiv \frac{1J}{C}$
- The electric field, normally expressed in units of Newtons/Coulomb may be also expressed conveniently in terms of volts/meter:

A unit of energy commonly used in physics is the *electron volt* (eV). An electron volt is the change in energy that an electron (or proton) acquires when moving through a potential difference of 1Volt.

$$1eV = 1.6 \times 10^{-19} C \bullet V = 1.6 \times 10^{-19} J$$

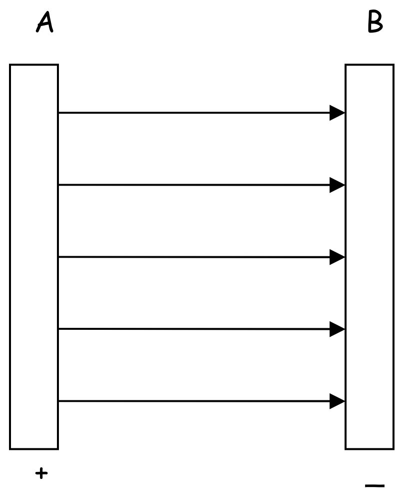
Example: The electrons that strike the screen inside of a typical TV set are traveling at a speed of 5 x 10^7 m/s. Through what potential difference must they be accelerated, from rest, in order to achieve this velocity.

$$\frac{1}{2}(9.1\times 10^{-31} kg)(5\times 10^{7}\, m/s)^{2} = \frac{1.1\times 10^{-15}\, J}{1.6\times 10^{-19}\, C} = 6.9 kV$$

So the storage capacitors in a typical TV set must be able to establish a potential difference of nearly 7000 volts.

Potential Differences in a Uniform Electric Field

Consider a *parallel plate capacitor*.



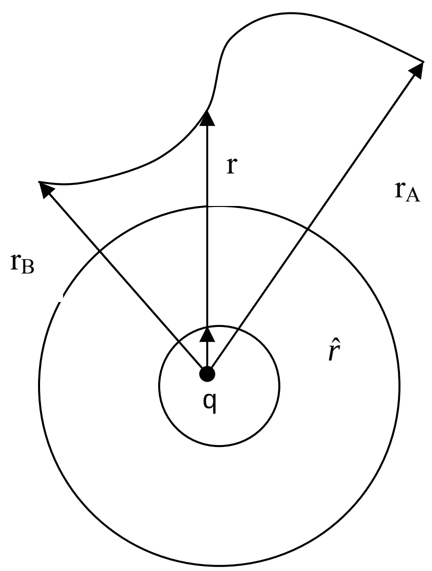
The field within is uniform and directed toward the right. The distance between the plates is *d*.

$$V_B - V_A = \Delta V = -\int_A^B \vec{E}\cdot d\vec{s} = -\int_A^B E\cos(0)ds = -\int_A^B Eds = -E\int_A^B ds = -Ed$$

$$V_{AB} = -Ed$$

- The minus sign indicates that plate B is at a lower potential than plate A, i.e., $V_B < V_A$.
- From this result it is apparent that for a positive test charge ΔU is negative.
- *A positive test charge loses electric potential energy when it moves in the direction of the electric field, i.e., along electric field lines of force.*
- Note that the integral in question vanishes if the path is perpendicular to the electric field.
- This path corresponds to movement along equipotential lines.
- It is important to remember that the result obtained here does not depend upon the path taken between A and B, only that the charge begins on plate A and ends up on plate B, i.e., is displaced in the direction of the electric field lines.

The Potential of a Point Charge



Consider an isolated positive charge, *q*, as shown at left. Recall that such a charge produces an electric field oriented radially away from the charge and that the equipotential surfaces are a series of concentric shells enclosing *q*. The path between A and B is *s*. We wish to compute the electrical potential between points A and B.

Recall $\vec{E} = k\frac{q}{r^2}\hat{r}$, this means that $\vec{E}\cdot d\vec{s}$ may be expressed as $k\frac{q}{r^2}\hat{r}\cdot d\vec{s}$.

Notice that any displacement, ds, produces a change dr in the magnitude or r and the result is that $\vec{E}\cdot d\vec{s} = k\frac{q}{r^2}dr$. Hence:

$$V_B - V_A = -\int_A^B E_r\,dr = -kq\int_{r_B}^{r_A}\frac{dr}{r^2} = kq\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

Notice that the result depends only upon the radial coordinates r_A and r_B. If A is taken to be infinity, then the potential at point B is:

$$V_B = -\int_\infty^B \vec{E}\cdot dr = k\frac{q}{r}$$

In general the potential at some point *P* due to a point charge is $V_P = k\frac{q}{r}$

The Potential of an Array of Point Charges

The electric potential of two or more point charges is merely the superposition of the individual potentials at a given point in space

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$$V_P = k\sum \frac{q_i}{r_i}$$

The potential energy of a system of two charged particles is:

$$U = k\frac{q_1 q_2}{r_{12}}$$

- Notice that if the polarity of both charges is the same, *U* is positive.
- This is consistent with the fact that like charges repel and so work must be done *on* the system in order to bring the charges together.

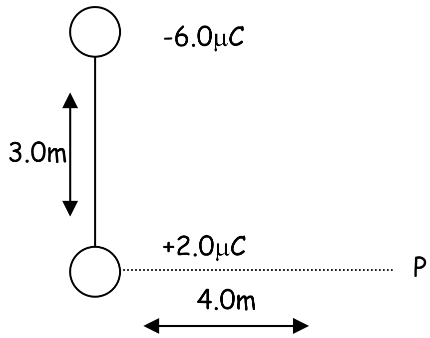
The potential energy of a system of three charged particles is:

$$U = k\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

- Imagine that q_1 is fixed at some position and that q_2 and q_3 are at infinity.
- The work required to bring q_2 from infinity to a position near q_1 is represented by the first term.
- The work required to bring q_3 from infinity to its position near q_1 and q_2 is represented by the last two terms.
- The result is independent of the order in which the charges are moved.

Example

Consider two charges as shown below.



Compute the potential at point P.

$$V_P = k\sum \frac{q_i}{r_i} \rightarrow = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

$$V_P = -6.30 \times 10^3 V$$

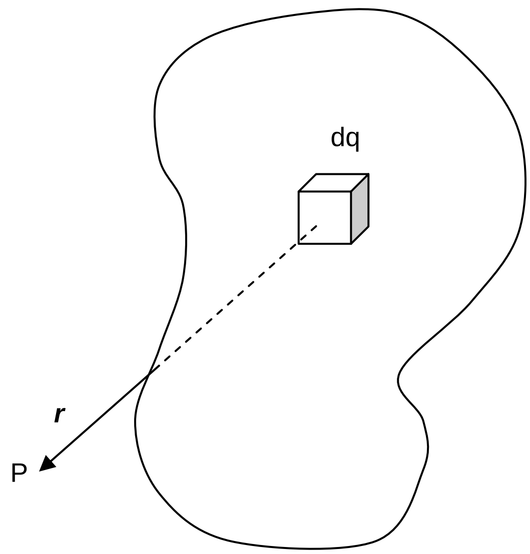
What is the potential energy of this system?

Electrical Potential Due to Continuous Charge Distributions

There are two methods used to determine the potential for an extended charge distribution.

The **first method** involves dividing up a know charge distribution into an infinitesimal number of dQ's, treating each like a point charge, them summing the individual potentials.

- This method works well for a known charge distribution.

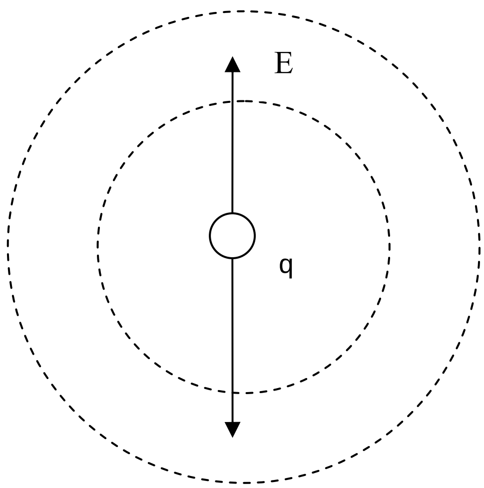


Consider the potential due to a small element dq and treat each dq as a point charge. For each dq $dV = k\frac{dq}{r}$

$\Rightarrow V = k\int \frac{dq}{r}$ Note: a reference point is implied here V = 0 for *P* at ∞.

The **second method** involves using the relationship between the electric field and electric potential $V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$.

- This method works well if the E field is known or may be evaluated in some relatively easy manner (Gauss's Law).
- This method yields potential difference. One may then take V_B to be 0 at any convenient point and solve for V_A.



Consider an isolated charge, +q. The electric field for this configuration is well known.

E is directed radially away from q and q is surrounded by a series of equipotential spheres

The magnitude of ***E*** depends only upon ***r***

$$\Rightarrow \vec{E} \cdot d\vec{s} = dV \rightarrow \vec{E}_r d\vec{r} = dV$$

$$\vec{E}_r = -\frac{dV}{d\vec{r}}$$

For this configuration potential changes in the radial direction only.

Example

Let's examine the potential of a uniformly charged sphere with the following properties:

-Solid Insulating Sphere
- Radius R
- Uniform charge density
- Charge Q+

What is the potential at point *P* outside the sphere

This is another example of a charge configuration with a well-known electric field so method 2 comes immediately to mind. From Gauss's Law:

For r > R (point *P*)

$$E_r = k\frac{Q}{r^2}$$

Since $\vec{E} \cdot d\vec{s} = E_r\, dr$

$$V_p = -\int_{\infty}^{r} E_r dr = -kQ\int_{\infty}^{r} \frac{dr}{r^2}$$

$$V_p = k\frac{Q}{r} \qquad \text{for r > R}$$

For r = R

Note that $V_p = \frac{kQ}{R}$ on the surface of the sphere as well. The potential is continuous across the surface while the electric field is not. There is a discontinuity of ***E*** but not V at the surface of the sphere.

For r < R (inside – point *D*)

Recall: $E_r = \dfrac{kQ_r}{R^3}$ (shown previously)

$$V_D - V_R = -\int_R^r E_r\,dr = -\frac{kQ}{R^3}\int_R^r r\,dr$$

↑ At radius of sphere (the surface)

↘ From Gauss's Law

$$= \frac{kQ}{2R^3} r^2 \Big|_r^R$$

$$= \frac{kQ}{2R^3} R^2 - r^2$$

Notice that we have to go from r to R using method 2 because of discontinuity in ***E*** at the surface of the sphere.

Using method 1 we could go from ∞ to r.

Note: $V_R = \dfrac{kQ}{R}$

$$V_D - V_R = \frac{kQ}{2R^3} R^2 - r^2$$

$$V_D - \frac{kQ}{R} = \frac{kQ}{2R^3} R^2 - r^2$$

$$V_D = \frac{kQ}{R} + \frac{kQ}{2R^3} R^2 - r^2$$

$$V_D = \frac{kQ}{R} + \frac{kQ}{2R}\left(1 - \frac{r^2}{R^2}\right)$$

$$V_D = \frac{kQ}{R} + \frac{kQ}{2R} - \frac{kQr^2}{2R^3}$$

$$V_D = \frac{2kQ}{2R} + \frac{kQ}{2R} - \frac{kQr^2}{2R^3}$$

$$V_D = \frac{kQ}{2R}\left(2 + 1 - \frac{r^2}{R^3}\right)$$

$$V_D = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^3}\right) \qquad \text{for } r < R$$

Note that at r = R on the surface of the sphere $\frac{kQ}{R} = V_R$ which is consistent with the previous result

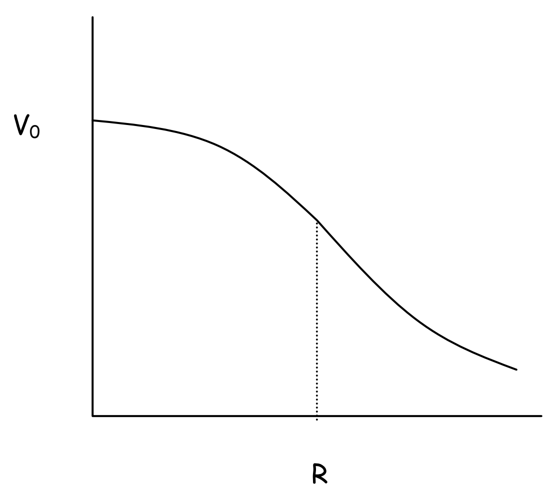
At the center of the sphere **(r = 0)**

E = 0

$$V_o = \frac{kQ}{2R}(3 - 0)$$

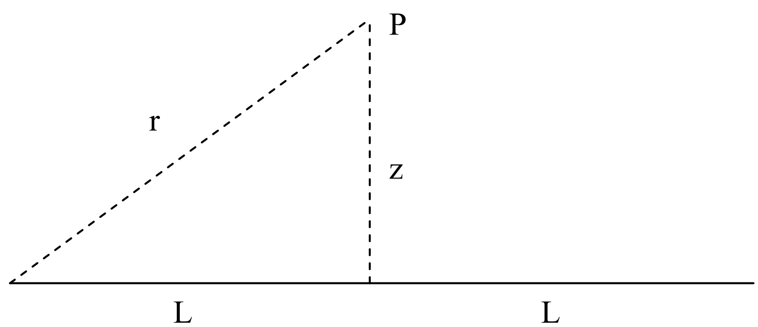
$$V_o = \frac{3kQ}{2R}$$

So the potential has its maximum value at r = 0 and diminishes as one moves away from the center of the sphere.



Example

Compute the potential of a finite line charge Q, length 2L, at point P.



Here we'll use method 1.

$$V_P = k\int \frac{dQ}{r}$$

Note: dQ = λdL, $r = \sqrt{z^2 + L^2}$

$$V_P = 2\frac{\lambda}{4\pi\varepsilon_o}\int_0^L \frac{dL}{\sqrt{z^2 + L^2}}$$

changing the limits

$$V_P = \frac{\lambda}{2\pi\varepsilon_o} ln\left(L + \sqrt{L^2 + z^2}\right)\Big|_0^L$$

$$V_P = \frac{\lambda}{2\pi\varepsilon_o}\left[ln\left(L + \sqrt{L^2 + z^2}\right) - ln\sqrt{z^2}\right]$$

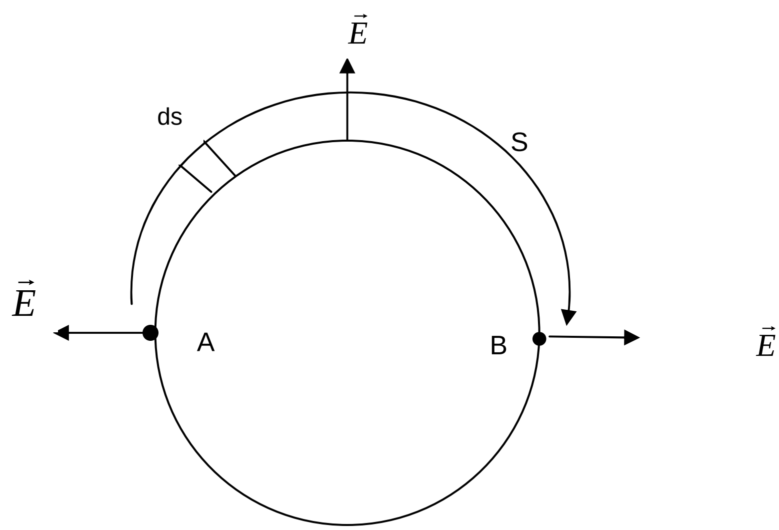
$$V_P = \frac{\lambda}{2\pi\varepsilon_o} ln\frac{L + \sqrt{L^2 + z^2}}{z}$$

The Potential of a Charged Conductor (proof that the surface of a conductor is an equipotential surface)

Consider a charged conductor in equilibrium. We have previously determined:

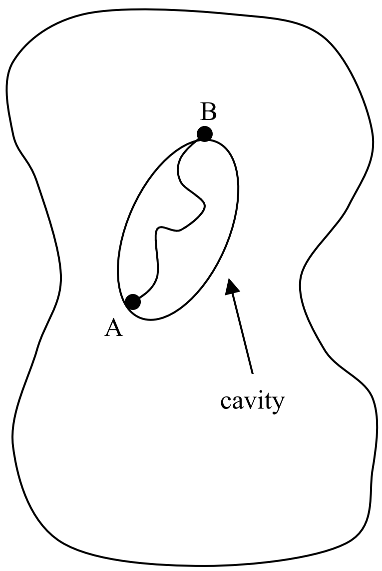
- Charge resides on the outer surface
- $\boldsymbol{E}_{outside}$ is perpendicular everywhere to the outside surface of the conductor
- $\boldsymbol{E}_{inside}$ = 0 everywhere

Now consider two points, A and B, on the surface of the conductor.



- Notice that along the surface connecting A and B $\boldsymbol{E}$ is always $\perp$ to ds
  $\therefore \vec{E} \cdot d\vec{s} = 0$
- This means that $V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s} = 0$ which implies that A + B are at the same potential.
- The surface of a conductor is an equipotential surface

A Cavity Within a Conductor



No charge inside the cavity

Show that $\vec{E}$ inside the cavity is zero.

Recall: every point on the surface (inner or outer) of the conductor is at the same potential so any two points on the surface of the cavity must be at the same potential, e.g., $V_A = V_B$.

$$V_B - V_A = 0 = -\int_A^B \vec{E} \cdot d\vec{s} \rightarrow E = 0$$

- The field inside must be zero. Can you make an argument from a geometric perspective to support the mathematical argument?
- The electric field inside any hollow conducting surface is zero
- Surrounding any cavity with conducting material suppresses electric fields inside.
- Faraday cage

Getting the Electric Field from Potential

Recall our potential function $V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$.

This already relates V and ***E*** and as we have shown previously $dV = -\vec{E} \cdot d\vec{s}$

If $\vec{E}$ has only 1 component e.g. E_x then $\vec{E} \cdot d\vec{s} = \vec{E}_x d$ and $\vec{E}_x = -\frac{dV}{dx}$

Note: V changes only along the coordinate x. No change $\perp$ to x (an equipotential surface)

In general, V(r) is a function of R^3 in rectangular coordinates:

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

Partial Derivatives

Example: If $V = 3x^2 + y^2 + yz$ then:

$$\frac{\partial V}{\partial x} = 6x \qquad \frac{\partial V}{\partial y} = 2y + z \qquad \frac{\partial V}{\partial z} = y$$

Introduce vector notation: $E = -\nabla V$ where $\nabla \equiv \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$

- ∇ (Del) is known as the *gradient* operator.
- The gradient function computes partial derivatives
- The gradient is a function that converts a scalar value into a vector.

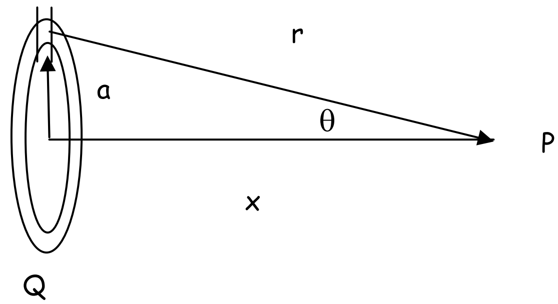
If *V* = 3xz + 2y

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \right)$$

$$\vec{E} = 3z\hat{i} + 2\hat{j} + 3x\hat{k}$$

Example

Consider a ring of charge, radius *a*, as shown below. Compute the electric potential at point *P*.



$$V = \frac{1}{4\pi\varepsilon_o}\int\frac{dQ}{r} = \frac{dQ}{\sqrt{x^2+a^2}}$$

$$V = \frac{k}{\sqrt{x^2+a^2}}\int dQ = \frac{kQ}{\sqrt{x^2+a^2}}$$

Note: If $a = 0$, V = kQ/x

What is the electric field at point P?

Due the symmetry, $\boldsymbol{E} = \boldsymbol{E}_x$:

$$E_x = -\frac{dV}{dx} = -kQ\frac{d}{dx}\left(x^2+a^2\right)^{-1/2} = -kQ\left(-\frac{1}{2}\right)\left(x^2+a^2\right)^{-3/2}(2x) = \frac{kQx}{\left(x^2+a^2\right)^{3/2}}$$

In agreement with the result obtained from direct integration of Coulomb's law.