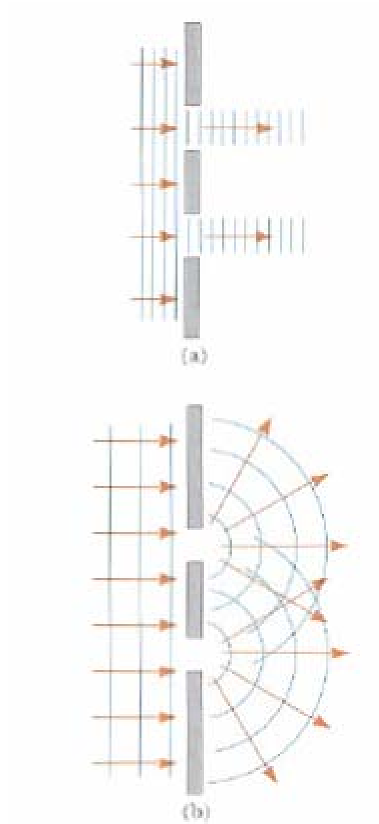
Diffraction

Diffraction may be thought of as a deviation of a light beam caused by partial obstruction of a wave front



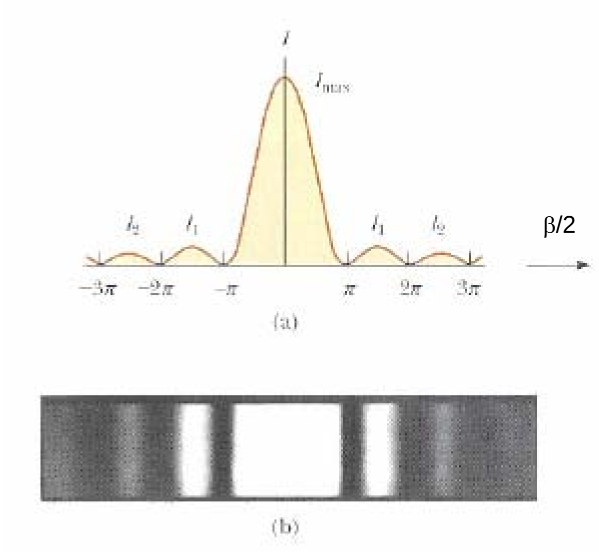
- Recall that interference effects depended on light spreading out after a wavefront passed through a slit.
- The spreading out is caused by partial obstruction of a wavefront by a mask or a sharp edge.
- Sound waves are diffracted in the same manner as light waves.
- Diffraction patterns are marked by a rapid decrease in intensity with increasing distance from the center of the pattern.

Fraunhofer diffraction

"far field" diffraction – assumption is that all light rays are approximately parallel

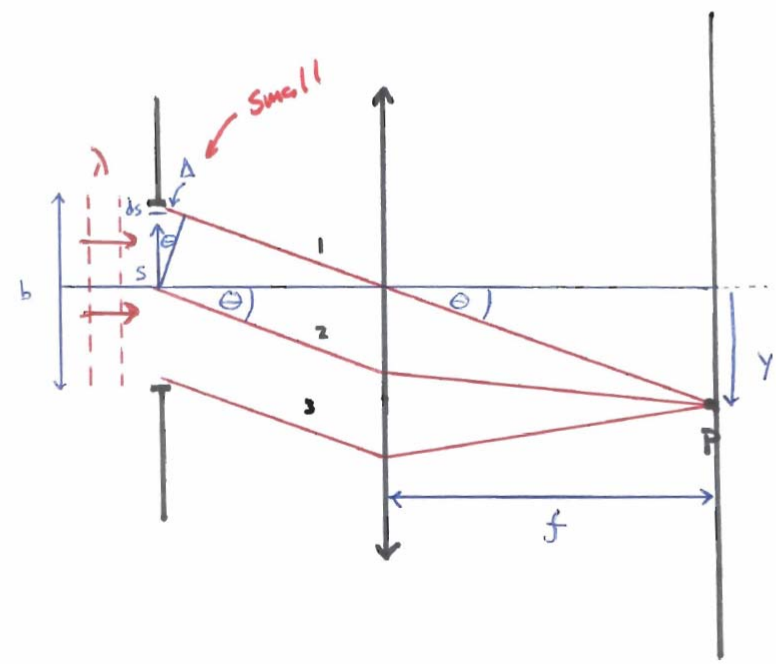
Single-Slit Diffraction

Consider Fraunhofer diffraction through a slit of width *b* as shown below. Note that the locations of relative maxima are about halfway between the minimum values (zero).



(a)

(b)



$$E_R = E_1 + E_2 + E_3$$

$$= \int dE_R$$

Consider each interval of ds as a source

It may be shown that the intensity at point *P* on the screen is given by:

$$I = I_0 \left[\frac{\sin \beta}{\beta} \right]^2$$

where $\beta = \frac{\pi}{\lambda} b \sin \theta$

(Note: some books use $\beta/2$ in the above expression in which case $\beta = \frac{2\pi}{\lambda} b \sin \theta$.

I_0 is the intensity at $\theta = 0$ or the position on the screen even with the center of the slit known as the *central maximum*.

Minima occur in the pattern at:

$$b \sin \theta = m\lambda$$

or

$$\sin \theta = m \frac{\lambda}{b} \qquad m = \pm 1, 2, 3, \ldots$$

Circular Apertures

Fraunhofer diffraction by a circular aperture is of great interest because most optical systems have round (rather than square) apertures.

- Maxima and minima form concentric rings.
- The central bright maxima is known as the *Airy Disk*.
- The mathematical expression for diffraction by a circular disk is more complicated:

$b \sin\theta = m\lambda$ (slit) must be replaced by

$b \sin\theta = J\lambda$ where *J* is a first order Bessel Function.

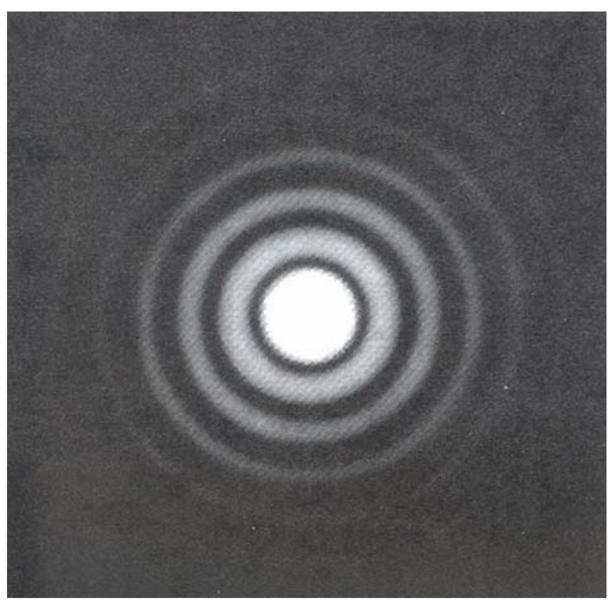
Positions of minima in Fraunhofer Diffraction:

	Slit	Circular
First Order	m = 1	J = 1.220
Second Order	m = 2	J = 2.233
Third Order	m = 3	J = 3.238

For diffraction with a circular aperture the Bessel Functions also oscillate between maxima and minima but decrease in amplitude with increasing distance from the central axis.

Positions of maxima: $J_1 = 1.635$
$J_2 = 2.679$

Diffraction behind a circular aperture is nearly the same as diffraction behind a circular obstacle.



Rayleigh's Criterion

Diffraction at a circular aperture sets the limits of resolution for virtually any optical system.

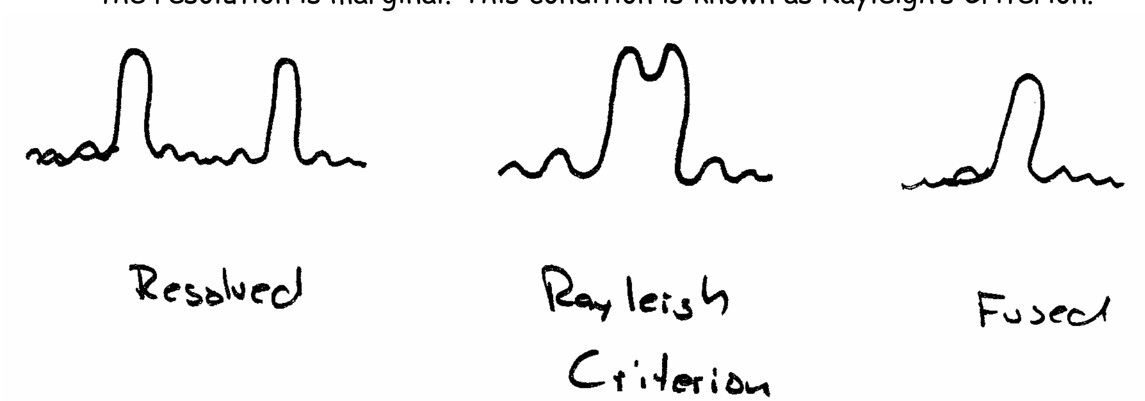
The Bessel functions for circular diffraction yield the diffraction limit for circular apertures:

$$\theta_{\min} \approx 1.22\frac{\lambda}{d}$$

Where θ_{min} is the angular separation and *d* is the diameter of the circular opening.

Consider a telescope aimed at two stars next to each other of approximately equal magnitude. In order to "resolve" the two stars, the diffraction patters of each in the focal plan of the telescope must be separate.

- When the central maxima fuse, the two stars appear as one.
- When the central max of one star coincides with the minimum of the other, the resolution is marginal. This condition is known as Rayleigh's Criterion.



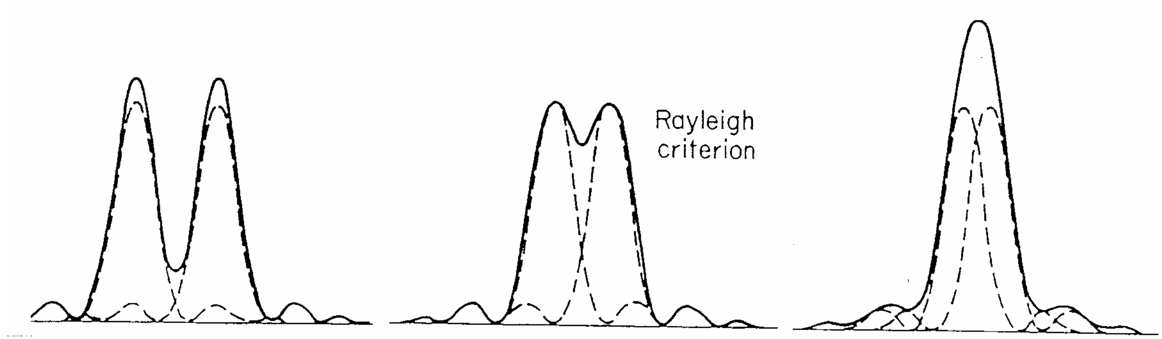
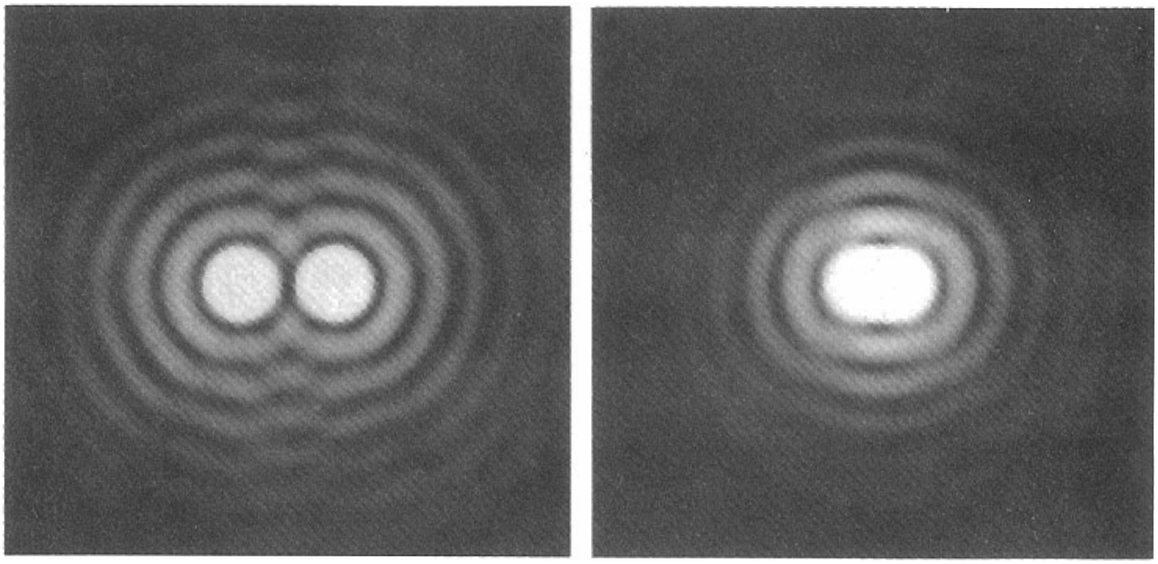


Figure 14-11 Rayleigh's criterion. Note the separation of the maxima in the left-hand plot and the close overlap on the right.



A well resolved image and an image at the limit of resolution (Rayleigh criterion)

Example Resolution of two stars 30,000 L.Y. distant in yellow light, with the eye.

$$\theta_{\min} = \frac{(1.22)(550nm)}{8 \times 10^{-3} m}$$

(diameter of a night adapted pupil)

$$\theta_{\min} = 8.39 \times 10^{-5} radians$$

Note: $s = r\theta$

$$= (30,000)(8.39 \times 10^{-5} rad)$$

$$s \approx 2.5\ L.Y.$$

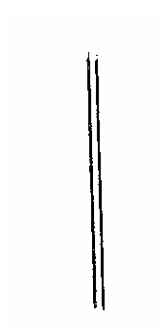
So, two stars must be 2.5 L.Y. apart to be resolved by the eye.

Diffraction Limit

- A lens, even one free of all aberrations, is still diffraction-limited.
- Diffraction limits apply to microscopes as well as telescopes.
- UV, x-ray and electron microscopes are used to extend the diffraction limit.
- Radio telescopes must be very large to have very good resolution for radio sources. Why?

The human eye is also diffraction limited

A test for visual Acuity



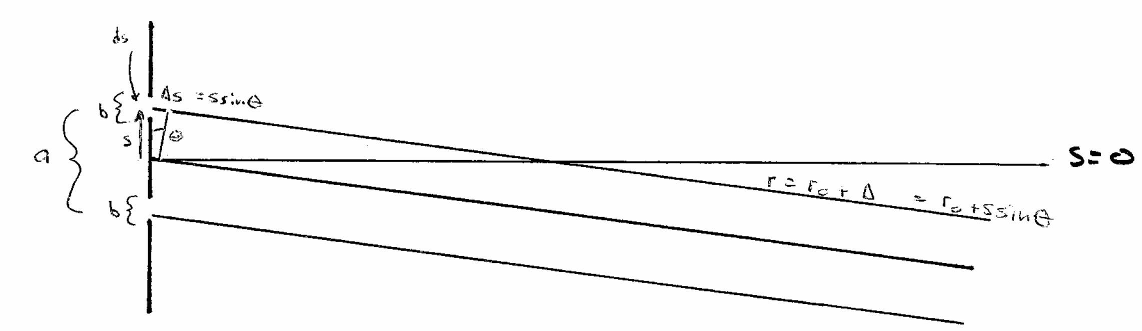
Can you resolve these two lines 1 mil apart at a distance of 4 meters? If so you have normal visual acuity.

At 12 inches, the normal visual acuity of the human eye is 0.00349 inch.

Double Slit Diffraction

Wavefronts are obstructed everywhere except at the slits

When we looked at two slit obstruction of a wavefront previously we assumed that the apertures (slits) were point sources. In reality, we must take into account the finite size of the slits.



It may be shown that the intensity at any random point on a far screen is determined by both interference and diffractive effects.

Recall that for two beam interference:

$$I = 4I_0 \cos^2\left(\frac{\pi}{\lambda} a \sin\theta\right) = 4I_0 \cos^2\left(\frac{\pi}{\lambda} a \frac{y}{s}\right)$$

and for single slit diffraction:

$$I = I_0 \left[\frac{\sin\beta}{\beta}\right]^2$$

For double slit diffraction it may be shown that:

$$I = 4I_0 \cos^2(\alpha)\left(\frac{\sin\beta}{\beta}\right)^2$$

where:

$$\alpha = \frac{\pi d \sin\theta}{\lambda} \quad \text{and} \quad \beta = \frac{\pi}{\lambda} b \sin\theta$$

Hence:

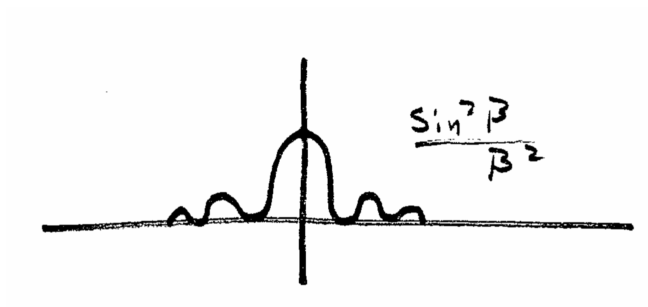
$$I = 4I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\left(\frac{\sin\left(\pi b \sin\frac{\theta}{\lambda}\right)}{\pi b \sin\frac{\theta}{\lambda}}\right)^2$$

where *b* is the slit width and *d* is the distance between the slits.

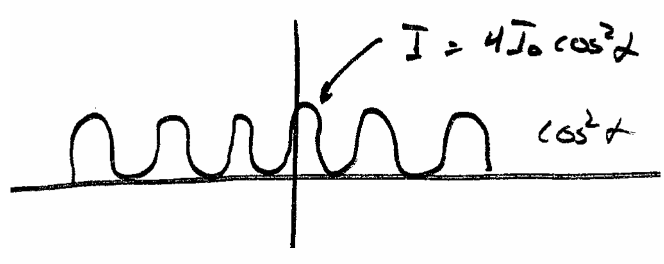
Note that the intensity is 4 times (in the center maximum) that of a single slit.

This expression looks very complicated but is fairly simple to dissect consisting of:

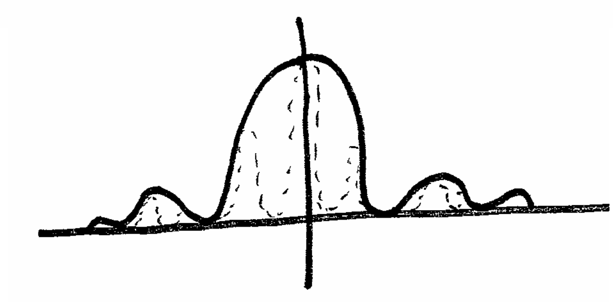
a diffraction term: $\left(\frac{\sin\beta}{\beta}\right)^2$

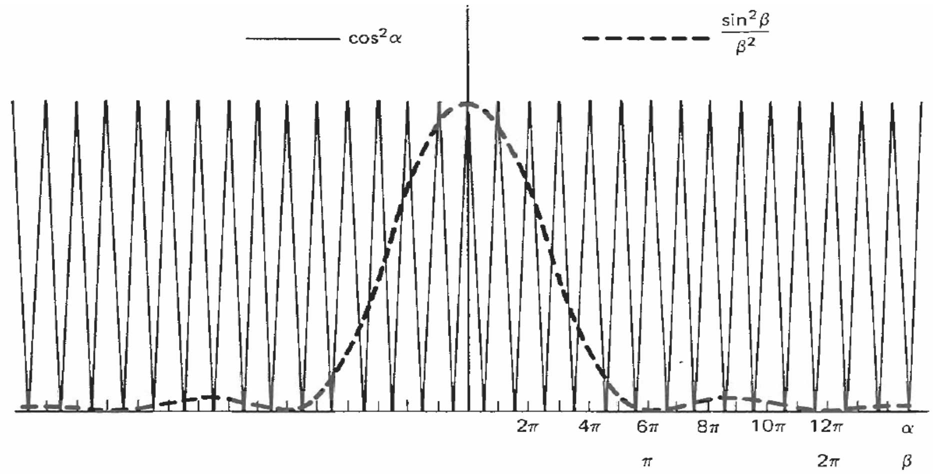


an interference term: $\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$

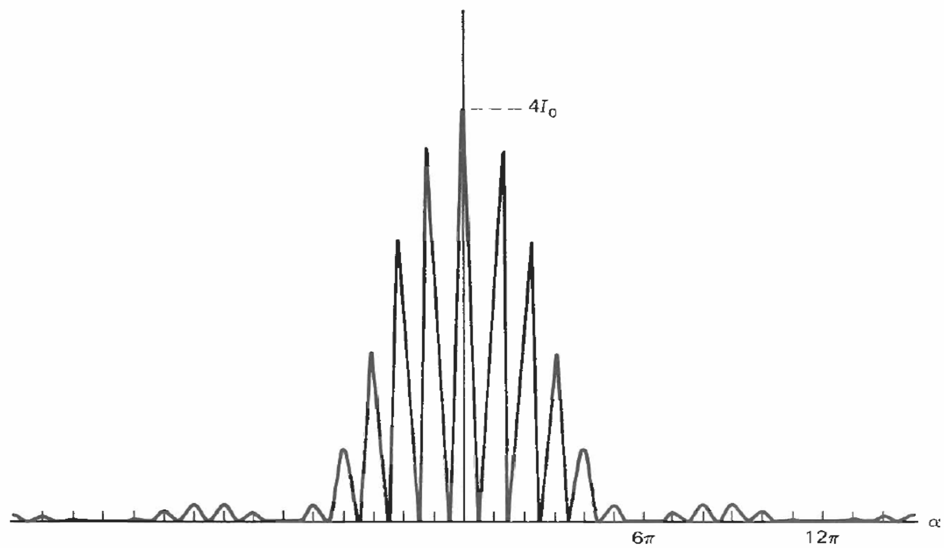


which combine to produce:

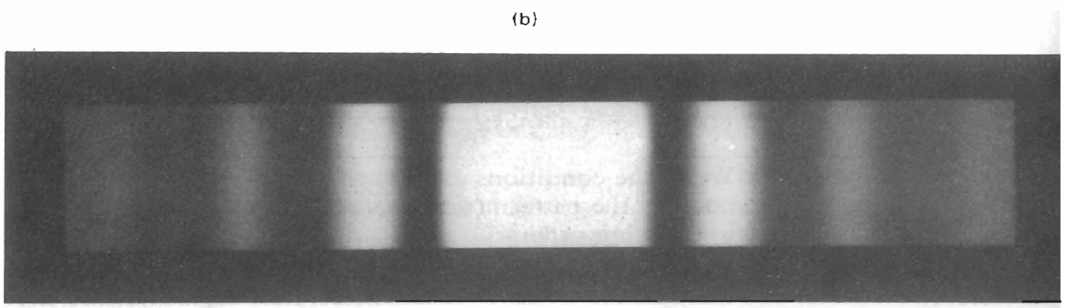



$\cos^2\alpha$
$\frac{\sin^2\beta}{\beta^2}$
2π 4π 6π 8π 10π 12π α
π 2π β

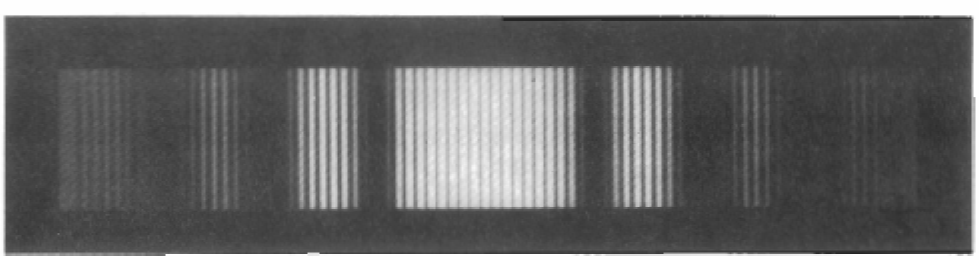
(a)


$4I_0$
6π 12π α

(b)



(c)



(d)

The diffraction envelope has minima when:

$$\beta = m\pi \qquad m = \pm 1, \pm 2, \ldots$$

or in terms of θ:

$$m\lambda = b\sin\theta \qquad \text{(diffraction minima)}$$

Notice that when these diffraction minima correspond to interference maxima the fringe is missing from the pattern.

Interference maxima occur when

$$\alpha = n\pi \qquad n = \pm 1, \pm 2, \ldots$$

or in terms of θ:

$$n\lambda = d\sin\theta \qquad \text{(interference maxima)}$$

When: $m\lambda = b\sin\theta$ and $n\lambda = d\sin\theta$ are satisfied for the same value of θ:

$$d = \frac{n}{m}b \text{ or } \alpha = \frac{n}{m}\beta$$

When d (slit separation) is an integer multiple of b (slit width) this condition is met exactly, e.g. $d = 2b \rightarrow n = 2m = \pm 2, 4, 6 \ldots$

- n must have an integer value so not all d/b rations work
- d/b must be > 1 or the physical implication is that the slit width is greater then the separation between them (impossible)
- when $\frac{d}{b} = \frac{4}{3} \rightarrow n = 4\frac{m}{3}$ so every 4th order will be missing coinciding with every 3rd envelope in the diffraction pattern

Multiple Slit Diffraction

Combination of previous methods (with lots of work) yields (for N identical equally spaced slits):

$$I = I_0 \left(\frac{\sin N\alpha}{\sin \alpha} \right)^2 \left(\frac{\sin \beta}{\beta} \right)^2$$

We are generally interested in very large values of N (*diffraction gratings*) where maxima are bright, distinct, and spatially well separated.

In this case it may be shown that intensity maxima are given by:

$$m\lambda = d \sin \theta \qquad \text{(the grating equation)}$$

- Many spectrometers use diffraction gratings instead of prisms to separate light into various spectral components.
- The resolving power, ***R***, of a grating is given by:

$$R = \frac{\lambda_{ave}}{\Delta\lambda}$$

- Gratings with high resolving powers can separate spectral components that are very close together.
- If N number of slits in a grating are illuminated it can be shown that for the m^{th} order diffraction the resolving power is:

$$\boldsymbol{R} = Nm$$

- Resolving power increases (larger is better) with both the number of slits and order number (in a spectrometer the orders more distant from the central maximum have greater separation).
- Note that for $m = 0$, $\boldsymbol{R} = 0$ which is consistent for the central maximum of any diffraction pattern.

Example Diffraction gratings have an equally spaced number of slits and are rated in terms of slits per cm. A common number of slits per cm is 6000. If such a grating is used to view a beam of pure blue light (440 nm) where will the 1st, 2nd, and 3rd order maxima be located?

$$d = \frac{1}{6000} = cm = 1.667 \times 10^{-4} cm = 1667nm$$ (why did we take the inverse here?)

$$\sin\theta_1 = \frac{1\lambda}{d} = \frac{440nm}{1667nm} = 0.264 \rightarrow \theta = 15.3^0$$

$$\sin\theta_2 = \frac{2\lambda}{d} = \frac{880nm}{1667nm} = 0.528 \rightarrow \theta = 31.9^0$$

$$\sin\theta_3 = \frac{3\lambda}{d} = \frac{1320nm}{1667nm} = 0.792 \rightarrow \theta = 52.3^0$$

Notice that if we try to computer the 4th order:

$$\sin\theta_4 = \frac{4\lambda}{d} = \frac{1760nm}{1667nm} = 1.06$$

so the sine is undefined and the 4th order maximum does not exist.

What is the resolving power of this grating in the 2nd order?

$$\boldsymbol{R} = Nm = 6000(2) = 12{,}000$$

Will this grating be capable of resolving the sodium doublet (589.00nm and 589.59nm) in the second order?

$$R = \frac{\lambda_{ave}}{\Delta\lambda} = \frac{589.30nm}{0.59nm} = 999$$

So yes, the grating has plenty of resolving power.