Centripetal Forces

The general case of *curvilinear* motion results from *centripetal forces* or "center-seeking forces. In the present discussion we will restrict ourselves to the special case of *uniform circular motion* (i.e., the path of the particle under the influence of a centripetal force is a circle). Uniform circular motion is easier to analyze than general curvilinear motion because only one acceleration, the centripetal acceleration, is present. Recall that in general:

$$F_c = ma = m\frac{v^2}{r}$$

Think of uniform circular motion in terms of your personal experiences. What causes it? What are the associated centripetal forces?

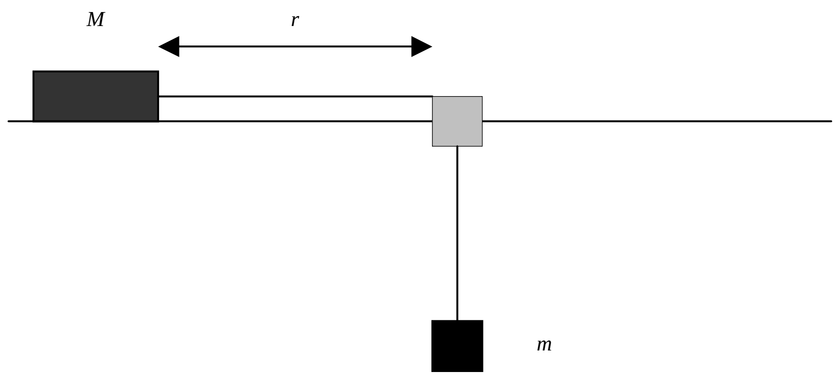
- The tension in a cord (twirling a rock on a string),
- Normal forces (a carnival ride)
- Static friction (tires on pavement)
- The force of gravity (planetary orbits)
- The Coulomb force (the motion of an electron about a nucleus).

- $F_c = m\frac{v^2}{r} = T$ Tension
- $F_c = m\frac{v^2}{r} = N$ Normal Force
- $F_c = m\frac{v^2}{r} = \mu_s N$ Static Friction
- $F_c = m\frac{v^2}{r} = G\frac{m_1 m_2}{r^2}$ Gravity
- $F_c = m\frac{v^2}{r} = k\frac{q_1 q_2}{r^2}$ Coulomb Force

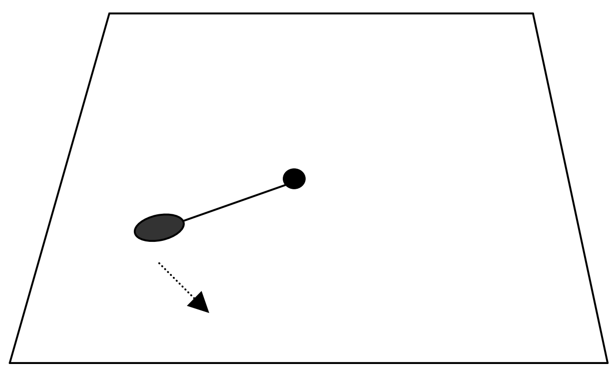
In general we wish to use one of the equations listed above account for centripetal forces.

It is usually unnecessary to use a Cartesian coordinate system when accounting for centripetal forces since, by definition, they point inward toward the center of rotation.

Example 1 Consider an air hockey puck on an air table (smooth) as shown below:



The puck (M) is attached via a light cord that runs through a hollow cylinder in the center of the table to a second block suspended freely beneath the table (m). The puck is set in motion and moves along a uniform circular path around the table with tangential velocity v.



If r is to remain constant what must the speed of the puck be and it's period of rotation?

First let's consider what happens if r is <u>not</u> kept constant.

- If the puck is at <u>rest</u>, the tension in the cord created by the hanging mass pulls the puck towards the hollow cylinder in the center of the table, decreasing r.
- If the puck circles the table at low speeds the tension created in the cord by the hanging mass is greater than the tension created by the motion of the puck and the puck moves toward the hollow cylinder in the center of the table, decreasing r at a more gradual rate than that above.

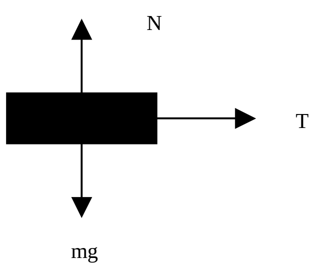
- If the puck circles the table at high speeds the tension in the cord created by the hanging mass is less than the tension created by the motion of the puck and the puck moves toward the outside of the center of the table, increasing *r*.

At this point it may be tempting for you to conjure up centrifugal force acting outward on the puck in order to account for the tension in the cord caused by the motion of the puck. This force does not exist! The outward force on this system acts on the cord, the cylinder at the center of the table, and ultimately the hanging block. The net force on the block must be centripetal (acting inward) or the block would continue to move in a straight line (in accordance with Newton's first law), not a curved path.

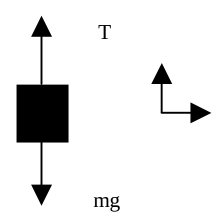
Remember that Newton's laws apply to *inertial* (non-accelerating) frames of reference. The tension in the cord does increase due to the circular motion of the puck, but not due to centrifugal force acting on the puck. The tension is created by the force that causes the puck to deviate from a straight line and is directed inward. The "outward" force in this system does not act on the puck.

Now let's consider what conditions are necessary for *r* to remain constant. The force created by the motion of the puck must be equal to the force on the hanging mass.

FBD puck:



FBD hanging mass:



$$\sum F_c = T = M\frac{v^2}{r}$$

$$\sum F_y = T - mg = 0$$

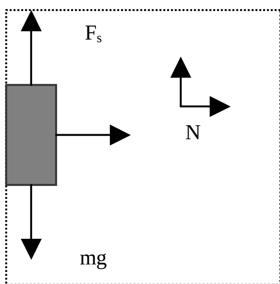
Note that *T* is the same in both equations (the hollow cylinder acts like a pulley in that it merely redirects *T*). Hence:

$$M\frac{v^2}{r} = mg \therefore v = \sqrt{\frac{rmg}{M}}$$

The *period* (the time it takes for one full circle of the puck around the table), τ, is distance divided by speed $\left(\frac{2\pi r}{v}\right)$, so: $\tau = \frac{2\pi r}{\sqrt{\frac{rmg}{M}}}$ seconds.

Example 2 *A "roto ride" at the fair consists of a large hollow cylinder (radius = 5 meters) capable of rotating at high speeds (0.5 rev/s). Riders enter the drum and stand around the edges of the cylinder with their backs to the wall. When the cylinder spins up to speed the floor is lowered and the riders are stuck to the walls. What is the minimum coefficient of static friction necessary to keep a person suspended?*

- In this case the force of static friction is responsible for keeping the person suspended but the normal force is responsible for the centripetal force.
- It will be necessary to convert some of the information given above to mks units to solve the problem



$$\sum F_c = N = m\frac{v^2}{r}$$
$$\sum F_y = F_s - mg = 0 \text{ (Equilibrium)}$$
$$\therefore \mu_s N = mg$$

Combining the two equations:

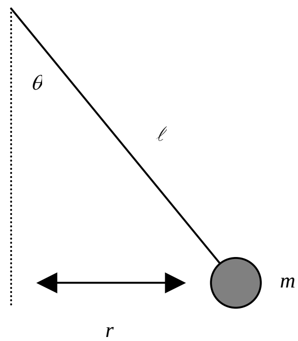
$$\mu_s m\frac{v^2}{r} = mg \rightarrow \mu_s \frac{v^2}{r} = g \therefore \mu_s = \frac{rg}{v^2}$$

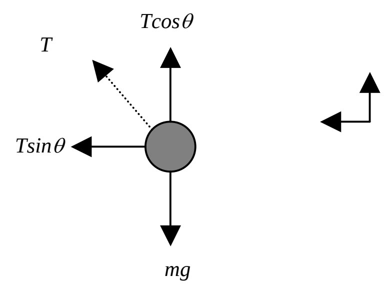
We were given the radius of the cylinder but what about the velocity? If the *frequency* ($1/\tau$) of rotation is 0.5 rev/s then:

$$\frac{0.5rev}{s} \times \frac{(2\pi r)meters}{rev} = 15.7m/s$$

Using this value yields μ_s **= 0.199**.

Example 3 Consider the conical pendulum shown below. The mass (*m*) follows a circular path around the dashed line at a constant speed such that the angle, θ, remains constant. Find the period, τ.





- This system works the same as twirling a rock attached to a cord around one's head, but with insufficient speed to make the make the entire system (cord and rock) orbit in a horizontal plane.

- In this case a component of the tension, *T*, in the cord is responsible for the circular motion and thus supplies the centripetal force.

- The "twist" here is that the Tension in the cord has vector components

$$\sum F_c = T\sin\theta = m\frac{v^2}{r}$$
$$\sum F_y = T\cos\theta - mg = 0 \therefore T\cos\theta = mg$$

Combining these two equations:

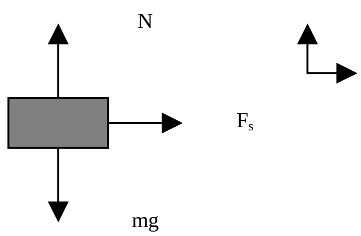
$\tan\theta = \frac{v^2}{rg}$ and since $r = \ell\sin\theta$, $\tan\theta = \frac{v^2}{\ell\sin\theta g} \rightarrow v = \sqrt{\ell g\tan\theta\sin\theta}$

Recall: $\tau = \frac{2\pi r}{v} \therefore \tau = \frac{2\pi r}{\sqrt{\ell g\tan\theta\sin\theta}}$

Example 4 Consider the case of a 2800kg truck moving along a curved section of an icy road where μ_s = 0.25. What is the maximum safe speed that the truck can negotiate a turn of radius 50 meters?

- In this case the force of static friction between the tires and the pavement supplies the centripetal force (can you surmise why static friction is involved rather than kinetic friction?).

- We are asked to determine the maximum speed for which the centripetal force equals the force of static friction.



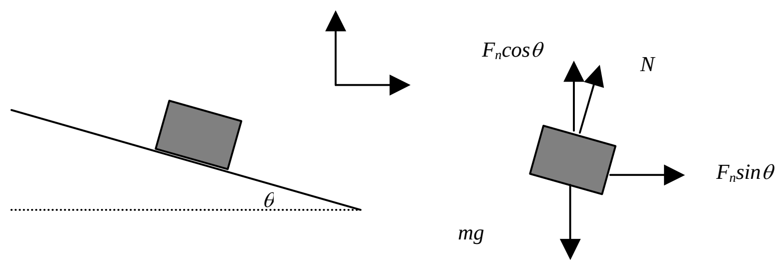
$$\sum F_c = F_s = m\frac{v^2}{r} \rightarrow \mu_s N = m\frac{v^2}{r}$$
$$\sum F_y = N - mg = 0 \therefore N = mg$$

Combining the two equations: $\mu_s g = \frac{v^2}{r}$ or $v = \sqrt{r\mu_s g}$ which for the given values yields **11.1 m/s** (about 25 mph).

By comparison, on dry pavement (μ_s = 0.60) the maximum safe speed is 17.1 m/s or about 38 mph.

Example 5 Highway curves are often *cambered* or banked to increase the maximum safe speed with which they may be negotiated in degraded circumstances (e.g., high speeds, slippery conditions, etc.). For a vehicle traveling at a speed *v* around a curve of radius *r*, determine the angle of banking such that frictional forces are not needed to keep the car moving along the radius of the curve (no sliding).

- In this case a component of the normal force between the tires and the pavement supplies the centripetal force.



- What is the significance of the non-rotated coordinate system used?

$$\sum F_c = F_n \sin\theta = m\frac{v^2}{r}$$
$$\sum F_y = F_n \cos\theta - mg = 0 \therefore F_n \cos\theta = mg$$

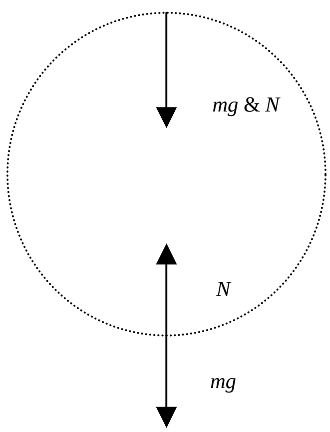
Combining these two equations yields $\tan\theta = \frac{v^2}{rg}$.

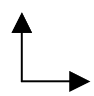
For a 100 kph interstate highway cloverleaf with a radius of 100 meters this yields an angle of 38^0. Clearly it is not practical to bank highways at this angle!

Example 6 So far we have considered uniform circular motion in a horizontal plane. Let's look at an example in a vertical plane where gravity plays a different role than that which we've considered previously.

Consider a fighter jet in a vertical loop. If the radius of the loop is 500 meters, what is the constant minimum speed required to maintain uniform circular motion at the top of the loop?

Here the centripetal force (F_c) is a normal force supplied by the air pressure acting on the wings and body of the aircraft. Since we are interested only in the top and bottom of the loop we will adopt our normal Cartesian reference for FBD's.





At the top of the loop:

$$\sum F_{top} = -mg - N = -m\frac{v^2}{r}$$

At the bottom of the loop:

$$\sum F_{bot} = N - mg = m\frac{v^2}{r}$$

At the top of the loop, at *minimum* speed the normal force goes to zero (why?). Hence: $mg = m\frac{v^2}{r} \therefore v = \sqrt{rg}$. This yields about **70 m/s** for the radius given.

How many "g's" is the airframe subjected to at the top of the circle at this speed?

$a_c = \frac{v^2}{r} = 9.8 m/s^2$ Which is **1 "g"** This result means that the centripetal force is equal to the force of gravity at the top of the arc. What sensation would the pilot feel?

How many "g's" is the airframe subjected to at the bottom of the circle at this speed?

$$N - mg = m\frac{v^2}{r} \therefore N = mg + m\frac{v^2}{r} \rightarrow a_t = a_g + a_c = 9.8m/s^2 + 9.8m/s^2 = 19.6m/s^2$$

Which is **2 "g's."** What sensation would the pilot feel?