Center of Mass

We wish to develop techniques for computing the center of mass for a system of particles and for computing the center of mass for extended rigid objects.

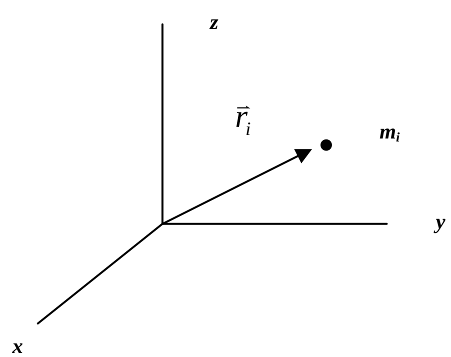
Center of mass principles:

1. A system moves as if all of its mass were concentrated at the center of the system.
2. The forces acting on a system act as if directed toward the center of mass of the system.

Center of mass for a system of particles:

$$r_c = \frac{\sum m_i r_i}{M}$$

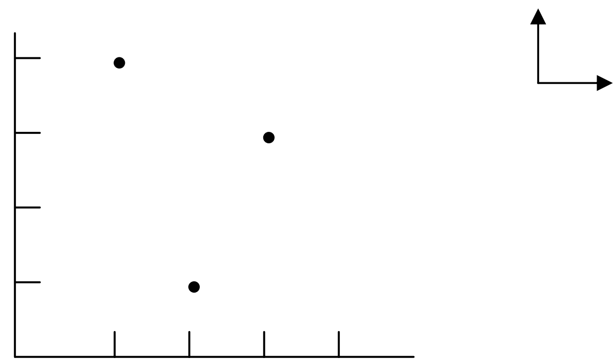
Where r_c is a general coordinate that may be replaced by x, y or z.



Center of mass for an extended rigid body:

$$r_c = \frac{1}{M}\int \vec{r}\,dm$$

Example 1. Find the center of mass for a three body system, m_1, m_2, m_3 = 3kg, as shown below. The tick marks are 1 meter increments.

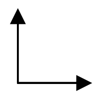


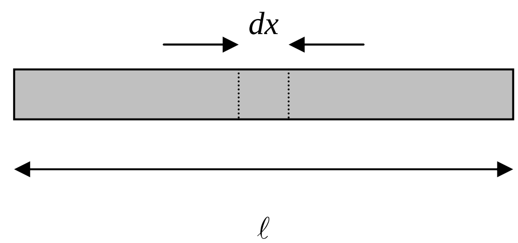
$$x_{cm} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{(1m)(3kg) + (2m)(3kg) + (3m)(3kg)}{3kg + 3kg + 3kg} = \frac{18m \cdot kg}{9kg} = 2.0m$$

$$y_{cm} = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3}{m_1 + m_2 + m_3} = \frac{(1m)(3kg) + (3m)(3kg) + (4m)(3kg)}{3kg + 3kg + 3kg} = \frac{24m \cdot kg}{9kg} = 2.7m$$

$(x, y)_{cm}$ = (2.0, 2.7) meters

Example 2. Find the COM for a rod of *uniform* linear density, λ, and length ℓ.





By definition, $r_c = \frac{1}{M}\int \vec{r}\,dm$

- This integral requires integrating with respect to mass rather than a spatial coordinate (*x, y, z*).
- It is possible to use the concept of linear density, λ (mass/length), which relates mass to length, to convert the mass in this integral into a spatial quantity
- $m = \frac{m}{\ell}\ell \rightarrow dm = \lambda dx$

Hence:

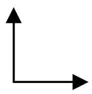
Definition in terms of *x*

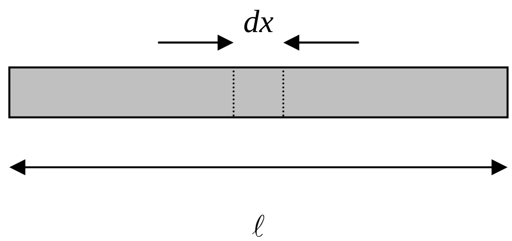
$$x_{cm} = \frac{1}{M}\int_0^{\ell} x\,dm = \frac{1}{M}\int_0^{\ell} x\lambda dx = \frac{\lambda}{M}\int_0^{\ell} x\,dx = \frac{\lambda x^2}{2M}\Bigg|_0^{\ell} = \frac{\lambda \ell^2}{2M}$$

Now if $\lambda = \frac{M}{\ell} \rightarrow x_{cm} = \frac{\frac{M}{\ell}\ell^2}{2M} = \frac{\ell}{2}$

Which means that the COM is at the center of the rod as one would expect for a uniform density.

Example 3. Find the center of mass for a rod of *non-uniform* linear density, $\lambda = \alpha x$ (where α is a constant of proportionality), and length ℓ.





This time the density is non-uniform ($dm = \lambda dx = \alpha x dx$) and must be integrated, hence:

$$x_{cm} = \frac{1}{M}\int_0^{\ell} x\,dm = \frac{1}{M}\int_0^{\ell} x^2 \alpha\,dx = \frac{\alpha}{M}\int_0^{\ell} x^2\,dx = \left.\frac{\alpha x^3}{3M}\right|_0^{\ell} = \frac{\alpha \ell^3}{3M}$$

The Motion of a System of Particles

The motion of a system of particles may be conveniently described by examining the motion of the center of mass of the system.

$$v_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{M}\sum m_i \frac{d\vec{r}_i}{dt} = \frac{\sum m_i v_i}{M}$$

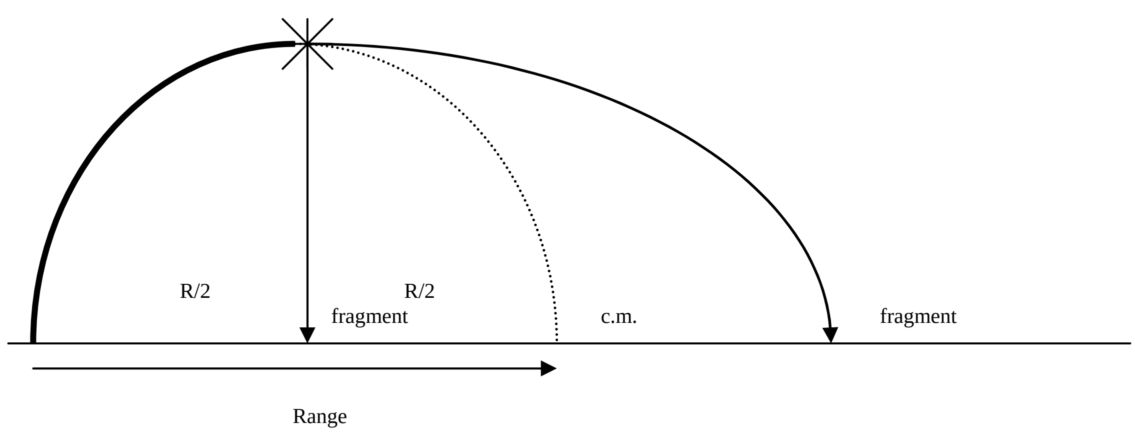
In terms of momentum:

$$M\vec{v}_{cm} = \sum m_i v_i = \sum p_i = \vec{p} \quad \text{(the total momentum of the system).}$$

This is the same value as the momentum of a single particle of mass M moving at velocity v_{cm}.

Since $F = \frac{d\vec{p}}{dt}$ the force acts as if it is concentrated at the center of mass of the system.

Consider the path of the exploding projectile shown below. If the projectile splits in half, and if momentum is to be conserved (which it is in the absence of external forces), the two fragments travel along the paths shown and the center of mass of the system continues along the original path.



Some of you may recall the popular (but very bad) film *Lost in Space*. The climatic scene involves the spaceship Jupiter II, which does not have enough power to achieve exit velocity, escaping an exploding planet by heading down through the center of the planet as it breaks up and emerging through the surface on the other side as the planet flies apart. Based on what you know about center of mass, is this a scheme likely to work outside of a film set?