Capacitance and Dielectrics

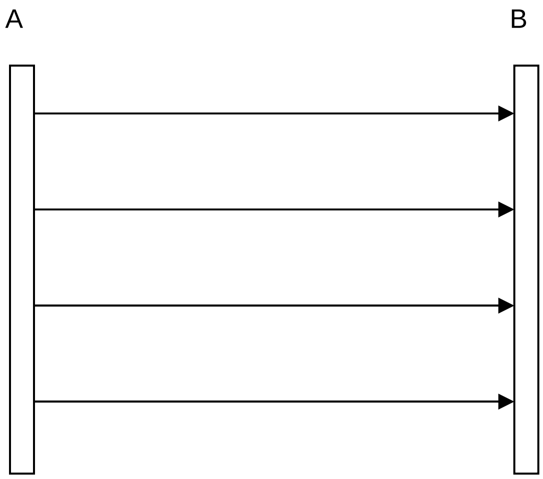
A *capacitor* is an electrical device that stores energy. Most electrical devices have some capacitance either intentional or accidental

- The energy is stored by maintaining a potential difference between two conductors.
- The energy is stored in the electric field between the conductors
- Any electrical device that creates electrical fields creates some capacitance
- The geometry of the capacitor affects the amount of energy stored
- Capacitors are rated by capacitance, *C*, which is the ability of a given capacitor to store charge.
- A *dielectric* is a material that when used in a capacitor increases the efficiency of the capacitor
- A capacitor is unlike a power supply (which also maintains an electrical potential between two points) in that it is limited by the fact that it can store only a finite amount of charge.
- We'll worry only about the positive charges stored in any capacitor since, by induction, the amount of negative charge is the same.

Capacitance

- Capacitance is the ratio of the magnitude of the charge in a capacitor to the magnitude of the potential difference in the capacitor
- $C \equiv \frac{Q}{V_B - V_A} = \frac{Q}{V} \rightarrow CV = Q$
- Capacitance is always a positive quantity
- Capacitance is a measure of the ability of a device to store charge and electrical potential energy.
- Units of capacitance: $\frac{Coulomb}{Volt} = 1 Farad$
- 1 Farad is a huge amount of capacitance.
- Typical capacitors have values on the order of 10^{-12} (picofarads) to 10^{-6} (microfarads)
- Capacitance depends upon the geometric arrangement of the conductors (plates).

The Parallel Plate Capacitor



If one ignores the fringing field the field between the plates is uniform. The distance between the plates is *d*.

- The field lines are parallel
- The field lines are perpendicular to each plate (why).

For an electric field between two parallel plates it may be easily shown that:

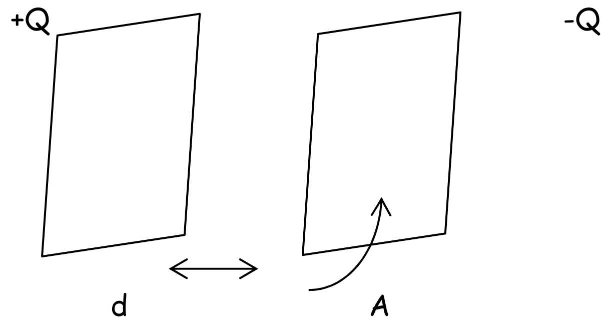
$$V_b - V_A = -\int_A^B \vec{E} \cdot d\vec{s} = -E\int_A^B \cos(0)ds = -Ed$$

where the minus sign indicates that plate B is at a lower potential than plate A, i.e., $V_B < V_A$.

Parallel plate capacitors are of enormous utility in physics and we will refer to them frequently.

Calculation of Capacitance

Parallel Plate Capacitor



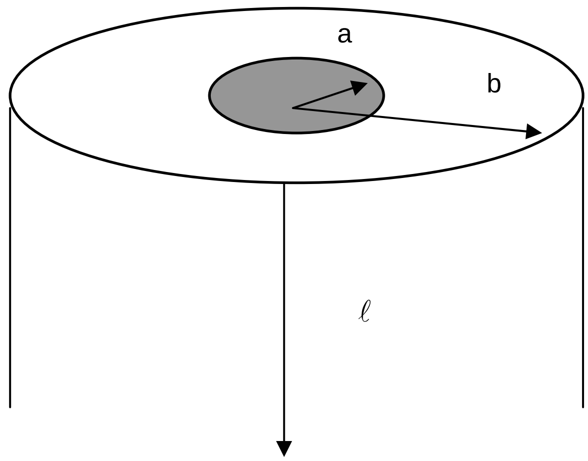
- The charge density for each plate is $\sigma = \dfrac{Q}{A}$
- We'll neglect fringe effects and assume uniform ***E*** field between the plates
- ***E*** = 0 everywhere outside the plates
- ***E*** between conducting plates is $\vec{E} = \dfrac{\sigma}{\varepsilon_o} = \dfrac{Q}{A\varepsilon_o}$
- The potential between the plates is $V_B - V_A = -\int \vec{E} \cdot d\vec{s} = -Ed$
- Ignoring the (-) sign:

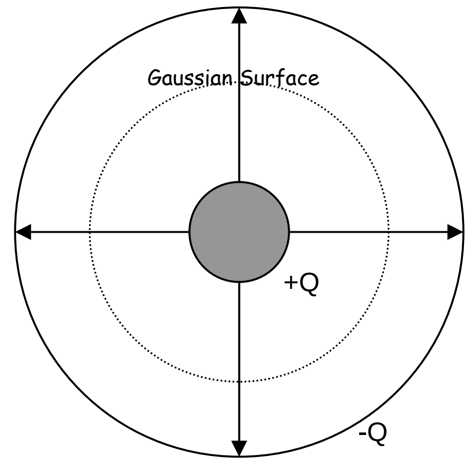
  $$V = Ed = \frac{Qd}{\varepsilon_o A} \rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\varepsilon_o A}} \rightarrow C = \frac{\varepsilon_o A}{d}$$

  for a parallel plate capacitor

- Note that capacitance for a parallel plate capacitor is proportional to area and inversely proportional to *d*.

Cylindrical Capacitor





Consider a cylindrical capacitor of length ℓ, inner radius *a*, outer radius *b*, with some charge Q.

- Assume ℓ >>> *a* or *b*
- Neglect fringing
- ***E*** is confined within the cylindrical plates

$$V_b - V_a = -\int \vec{E} \cdot d\vec{s}$$

Recall for a cylindrical charge distribution: $E = \frac{2k\lambda}{r} = \frac{\lambda}{2\pi\varepsilon_0 r}$

Note: this result applies here because the outer cylinder does not contribute directly to the electric field in the region of interest between the plates (draw a Gaussian surface outside and see what charge is enclosed).

$$V_b - V_a = -\int \vec{E}_r \cdot d\vec{r} = -2k\lambda \int_a^b \frac{dr}{r} = 2k\lambda \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{Q}{2k\lambda \ln\left(\frac{b}{a}\right)} = \frac{Q}{2k\frac{Q}{\ell}\ln\left(\frac{b}{a}\right)} = \frac{\ell}{2k\ln\left(\frac{b}{a}\right)}$$

Note that V_{ab} is a positive quantity because $2k\lambda \ln\left(\frac{b}{a}\right)$ is a positive quantity – this because *a* is at a higher potential than *b*.

The capacitance depends on geometry

- Higher values of ℓ yield higher capacitances
- The respective radii *a* and *b* affect capacitance

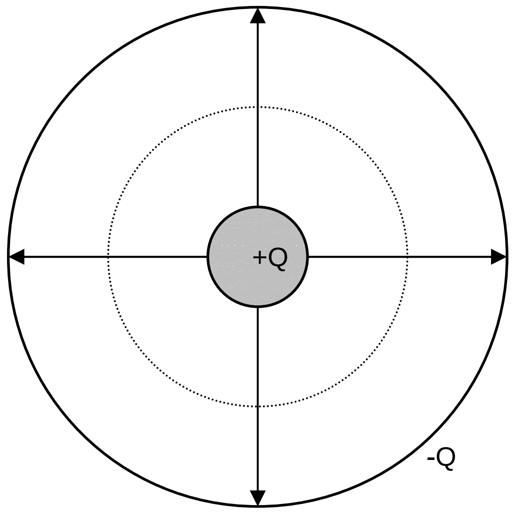
Coaxial Cable is a type of cabling that consists of two concentric cylindrical conductors of radii *a* and *b* separated by an insulator.

- The two conductors carry currents in opposite directions.
- Coaxial cable is widely used because this arrangement is very useful in shielding an electric signal from interfering fields external to the cable.
- An issue with coaxial cable (and in fact nearly all conducting wires) is capacitance per unit length.
- It is easy to show that the capacitance per unit length for coaxial cable is:

$$\frac{C}{\ell} = \frac{1}{2k\ln\left(\frac{b}{a}\right)}$$

Spherical Capacitor

Consider a spherical capacitor of inner radius *a*, outer radius *b*, with some total charge Q. Noting that the field between the plates is $\vec{E} = k\frac{Q}{r^2}\hat{r}$



$$V_b - V_a = -\int \vec{E}_r \cdot d\vec{r} = -kQ\int_a^b \frac{dr}{r^2} = kQ\left[\frac{1}{r}\right]_a^b = kQ\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$V_{ab} = kQ\left(\frac{b-a}{ab}\right)$$

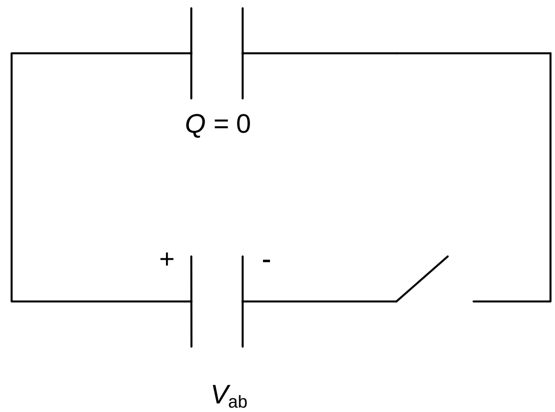
$$C = \frac{Q}{V} = \frac{ab}{k(b-a)}$$

Charging a capacitor

Consider the sequence of circuits shown.

t < 0. The switch is open and the source of electrical potential is isolated from the capacitor.

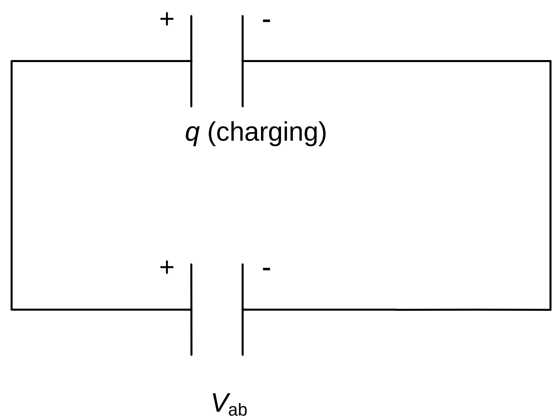
The capacitor is uncharged.



t = 0. The switch is closed and the capacitor begins to charge.

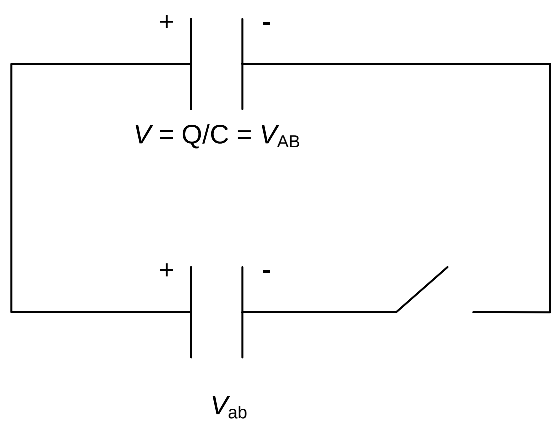
The instantaneous value of the charge on the capacitor while it is charging is some intermediate value *q* less than the final value Q of the fully charged capacitor.

The instantaneous voltage across the plates is *v* and grows with *q*.



t → ∞. When the capacitor is fully charged (Q) a steady state voltage of V is established across the plates.

The final potential V across the capacitor is equal to the applied potential.



Energy Stored in a Capacitor

- In order to examine the energy stored in a capacitor we'll assume an electrostatic situation for the capacitor.
- In order to do this we have to assume that the capacitor charges slowly through a series of quasi-static states.

In general: $V = \frac{Q}{C}$

$$V_f = \frac{Q}{C},\ V_i = 0$$

$$V_{AVE} = \frac{V}{2} = \frac{Q}{2C}$$

$$W = \frac{QV}{2} = \frac{Q^2}{2C} \qquad \text{(where } W = QV_{ave}\text{)}$$

The work done may be considered as increased potential energy stored in the capacitor.

$$U = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

- This result applies to an capacitor regardless of geometry
- The energy is stored in the field between the plates.

Alternatively consider a charge *q* on a capacitor at some point during the charging process. At the same moment the instantaneous potential across the capacitor is

$$v = \frac{q}{C}.$$

Now consider the work required to raise an increment of charge +dq from the low potential in the capacitor (normally zero) to whatever the maximum potential of the capacitor across the potential difference V_{ab}.

$$dW = Vdq = \frac{q}{C}dq \rightarrow W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C}$$

So the energy stored in the capacitor may be considered to be the same as the work done in moving a charge from low to high potential.

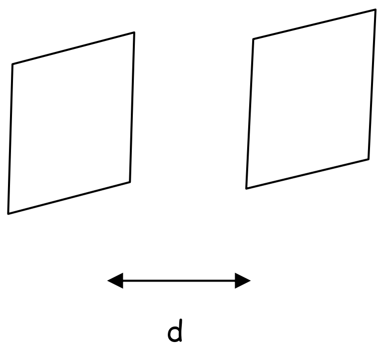
$$W = U = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

Energy Stored in a Parallel Plate Capacitor

$C = \frac{\varepsilon_o A}{d}$, $V = Ed$

$U = \frac{1}{2}\frac{\varepsilon_o A}{d}\left(E^2 d^2\right) = \frac{1}{2}\left(\varepsilon_o Ad\right)E^2$ (energy stored in a parallel plate capacitor)

- The volume of a parallel plate capacitor is its surface area times distance between the plates, or *Ad*

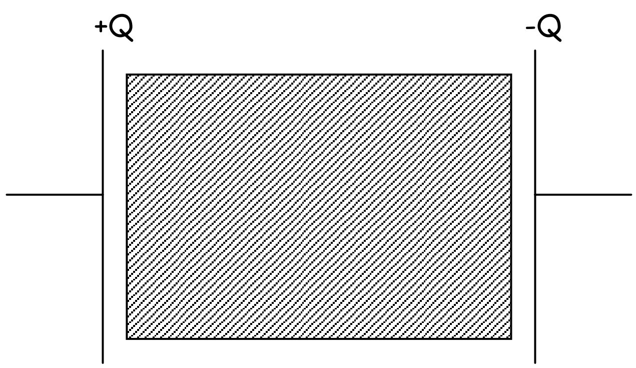


- Define energy density: $u = \frac{U}{Ad} \rightarrow u = \frac{1}{2}\varepsilon_o E^2$
- Notice that this result is independent of *A* or *d* and therefore applies to all capacitors, regardless of geometry.

Capacitors with Dielectrics

A dielectric is a partial conductor or a non-conductor. The addition of a dielectric to any capacitor increases the capacitance of a capacitor.

Capacitance for a normal parallel plate capacitor is $C = \frac{Q}{V}$. If the same capacitor is filled with a dielectric material (as shown below):

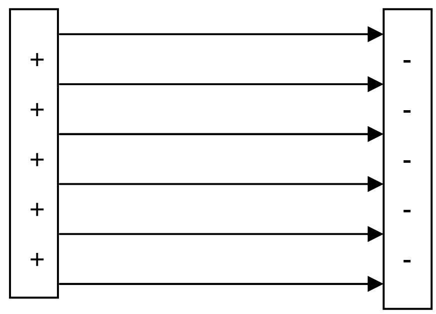


Dielectric Filled

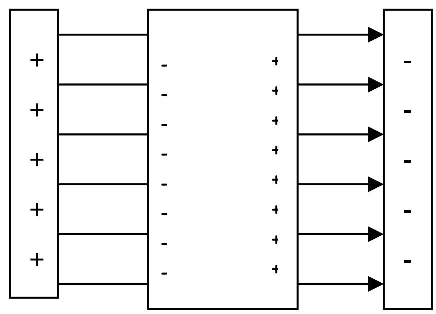
- V_D is observed to less than V
- Q remains constant
- Since $C = \frac{Q}{V} \rightarrow C_D > C$

Polarization of the dielectric material is the key to increasing capacitance.

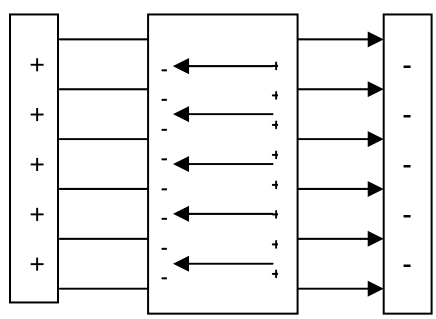
Before dielectric



After introduction of the dielectric material, the preexisting field polarizes the dielectric.



Polarization of the dielectric establishes a field within the dielectric that opposes the external field. Since electric fields are vector fields, the sum of these two fields results in a lesser net field within the capacitor.



Since $V = E d$, V is diminished with E.

Some properties of Dielectrics

- $C_D = C\kappa$, where κ is known as the *dielectric constant*
- $C = \kappa \dfrac{\varepsilon_0 A}{d}$
- permittivity of a dielectric is ε, permittivity of free space is ε_0
- $\dfrac{\varepsilon}{\varepsilon_o} = \kappa$
- $\varepsilon = \kappa\varepsilon_o$, ***E*** in an air-filled parallel plate = $\dfrac{\sigma}{\varepsilon_o}$, with a dielectric $\dfrac{\sigma}{\varepsilon}$.
- κ always > 1 in any material
- κ = 1 in a vacuum (free space), $\varepsilon = \varepsilon_0$
- $\kappa_{air} \approx 1$
- In an air/vacuum parallel plate capacitor $E = \dfrac{\sigma}{\varepsilon_o} = \dfrac{V}{d}$. Since *d* remains fixed, the reduction in *V* implies a reduction in ***E***.
- Dielectrics increase the maximum operating voltage of a capacitor.
- Dielectrics may increase the mechanical strength of a capacitor.

Energy Density in a Dielectric Filled Capacitor

$$u_0 = \frac{1}{2}\varepsilon_o E^2$$

$$u = \frac{1}{2}\kappa\varepsilon_o E^2 = \frac{1}{2}\varepsilon E^2$$

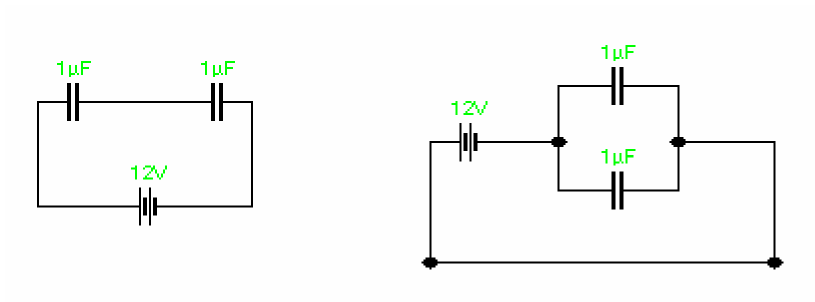
Dielectric Strength

- Whenever any insulator is placed in a strong enough $\vec{E}$ field it breaks down and begins to conduct, i.e., if one put a large enough potential difference across any insulator it will begin to conduct.
- The maximum $\vec{E}$ field a material can withstand without breakdown is its dielectric strength. Dielectric Strength is given in $\dfrac{V}{m}$.
- Tables of dielectric constants are found in most materials reference manuals.

Real Capacitors

- Typical small capacitors are made of metal foil with a layer of Mylar or paraffin-impregnated paper as the dielectric, rolled into a cylinder.
- Large, high voltage capacitors typically have several metal plates immersed in oil (sometimes containing PCB's).
- Ceramics are often used for very small capacitors.
- Variable capacitors consist of one fixed plate and one moveable plate generally with air as the dielectric.
- Electrolytic capacitors are used to store large amounts of charge at low voltages. Polarity – normally unimportant in capacitors, is important in the installation of electrolytic capacitors.

Series and parallel capacitive circuits

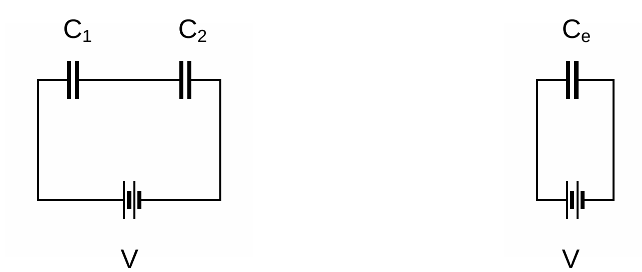


Series capacitive circuits

- The magnitude of charge is the same on all capacitors in series, i.e., $Q_1 = Q_2 = Q$. This seeming violation of conservation of charge occurs because power supplies may be thought of as an infinite supply of charge.

- The potential across capacitors in series must add up to the applied potential

- $1/C_{eq} = 1/C_1 + 1/C_2$ (to be shown)



capacitive circuit — equivalent capacitive circuit

Recall: $V = Q/C_{eq}$

$V_{app} = V_1 + V_2 \therefore Q/C_{eq} = Q/C_1 + Q/C_2 \therefore 1/C_{eq} = 1/C_1 + 1/C_2$ (QED)

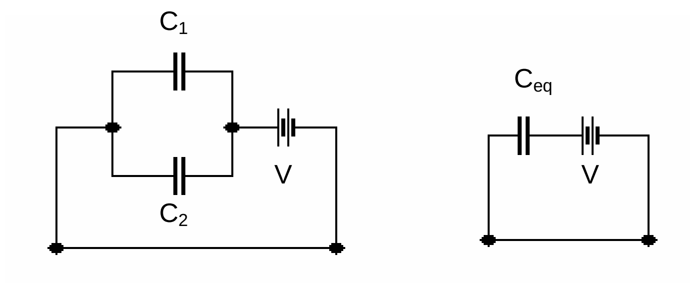
In general: $1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3 + \ldots 1/C_n$

Parallel Capacitive Circuits

- The potential across all capacitors in parallel is the same, and it is the same as the applied potential, i.e., $V_{app} = V_1 = V_2$.

- The sum of the charges on capacitors in parallel is the same as the total charge i.e., $Q_{tot} = Q_1 + Q_2$.

- $C_{eq} = C_1 + C_2$ (to be shown)



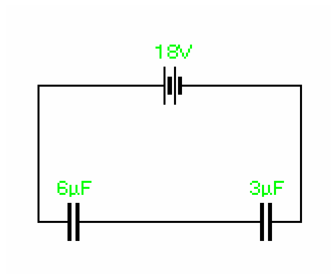
$Q_1 = C_1V, \quad Q_2 = C_2V, \quad Q_{tot} = C_{eq}V$

$C_{eq}V = C_1V + C_2V \therefore C_{eq} = C_1 + C_2$ (QED)

In general: $C_{eq} = C_1 + C_2 + C_3 + \ldots C_n$

Example Consider the circuit below. Compute the charge on each capacitor and the potential across each capacitor.

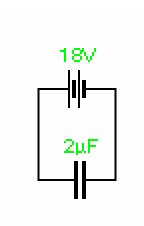
C_1 = 6μF, C_2 = 3μF, V_{app} = 18 V



1. Combine the two series capacitors:

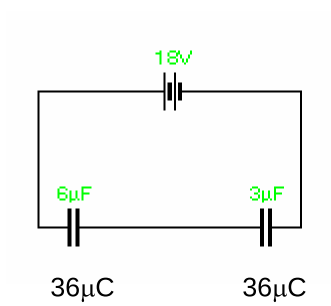
   $1/C_{eq} = 1/C_1 + 1/C_2 \Rightarrow 1/C_{eq} = 1/6\mu F + 1/3\mu F \therefore C_{eq} = 2\mu F$

   So the equivalent circuit contains a single 2μF capacitor.



2. Compute the charge on the equivalent capacitor:

   $Q_{eq} = C_{eq}V_{app} = (2\mu F)(18V) = 36\mu C$

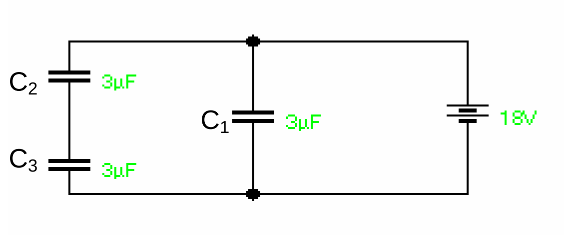


3. $V_1 = Q/C_1 = 36\mu C/6\mu F = 6$ V

   $V_2 = Q/C_2 = 36\mu C/3\mu F = 12$ V

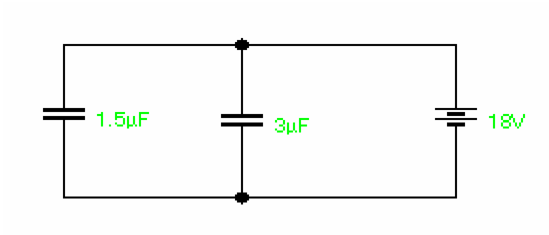
   Note: $V_1 + V_2 = V_{app} = 18$ V

Example Consider the circuit below. $C_1 = C_2 = C_3 = 3\mu F$, $V_{app} = 18$ V. Compute the charge on each capacitor and the potential across each capacitor.



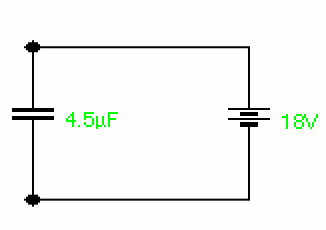
1. Find C_{23eq} and combine the series capacitors in the left branch of the circuit:

   $1/3\mu F + 1/3\mu F = 1/C_{eq} \therefore C_{23eq} = 1.5\mu F$



2. Combine the parallel capacitors and find the equivalent capacitor:

   $1.5\mu F + 3\mu F = 4.5\mu F$

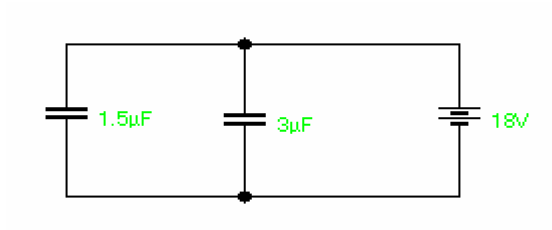


3. Find the charge on the equivalent capacitor:

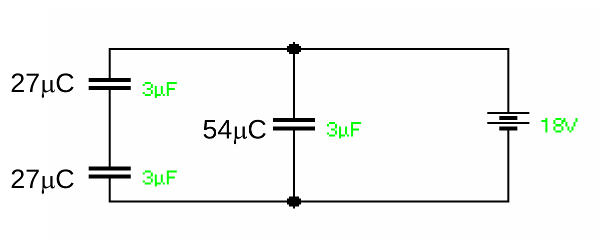
   $C_{eq} V_{app} = Q \Rightarrow (4.5\mu F)(18V) = 81\mu C = Q_{total}$

4. Find the charge on each capacitor given the potential across each capacitor (voltage is the same across parallel branches).

$C_1 V = Q \Rightarrow (3.0\mu F)(18V) = 54\mu C = Q_1$
$C_{23} V = Q \Rightarrow (1.5\mu F)(18V) = 27\mu C = Q_{23}$



5. Find the potential across each capacitor given the charge on each capacitor (charge is the same on all capacitors in series).



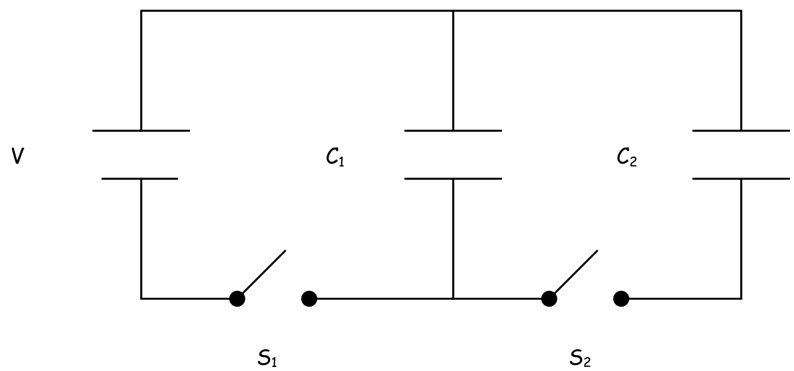
$V_2 = Q/C_2 \Rightarrow 27\mu C/3\mu F = 9$ V
$V_3 = Q/C_3 \Rightarrow 27\mu C/3\mu F = 9$ V

Notice that the sum of the voltages across each branch of the circuit is equal to the applied potential of 18 volts.

$C_1 \Rightarrow 3\mu F$, $54\mu C$, 18 V
$C_2 \Rightarrow 3\mu F$, $27\mu C$, 9 V
$C_3 \Rightarrow 3\mu F$, $27\mu C$, 9 V

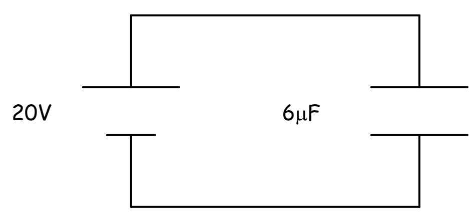
Conservation of Charge in Capacitor Networks

Consider the following capacitive network. Well charge the capacitors serially instead of all at once as follows.



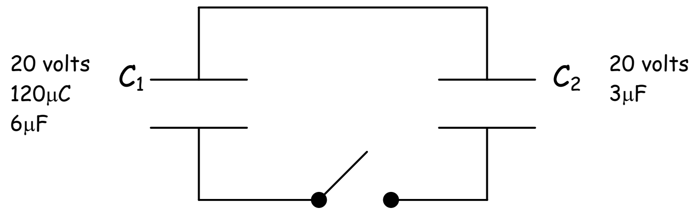
$$C_1 = 6\mu F$$
$$C_2 = 3\mu F$$
$$V = 20V$$

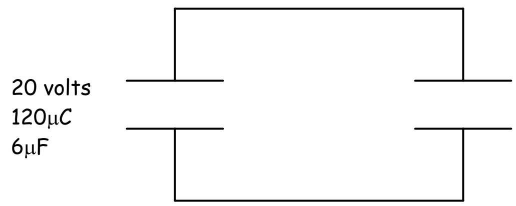
We'll close S_1, allowing C_1 to fully charge. Then we'll open S_1, close S_2, allowing C_2 to fully charge. This arrangement does not allow us to draw infinite charge from the power supply.



$$CV = Q \rightarrow (6\mu F)(20V) = 120\mu C$$

Now that C_1 is fully charged we'll isolate it from the power supply by opening S_1. We'll then use C_1 to charge C_2. Since C_2 holds a finite amount of charge (120μC) we have to do a little more work to compute the amount of this that will be transferred to C_2.





Conservation of charge requires that $Q_1 = 120\mu C - Q_2$

$$\frac{Q_2}{C_2} = V_{app} = \frac{Q_1}{C_1} \Rightarrow \frac{Q_2}{C_2} = \frac{120\mu C - Q_2}{C_1}$$

$$\frac{120\mu C - Q_2}{6\mu F} = \frac{Q_2}{3\mu F} \rightarrow \frac{360\mu C - 3Q_2}{18\mu F} = \frac{6Q_2}{18\mu F}$$

$$360\mu C - 3Q_2 = 6Q_2 \rightarrow 360\mu C = 9Q_2 \rightarrow 40uC = Q_2$$

Since Q_2 is 40μC

$$Q_1 = 120\mu C - Q_2 = 80\mu C$$

So Q_2 is 40μC and Q_1 is 80μC.