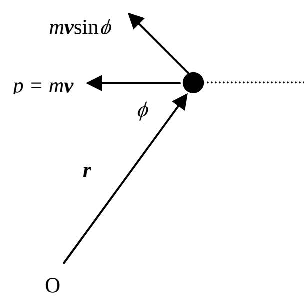
Angular Momentum

Just as moment of inertia is the rotational analog to mass, and torque is the rotational analog to force, *angular momentum* is the rotational analog to linear momentum.



- The angular momentum of a particle, ℓ, with respect to the origin O is: $\ell = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$.
- The product $\vec{r} \times \vec{p}$ is in a plane perpendicular to the plane containing ***r*** and ***p*** and in this case is out of the plane of the page.
- Angular momentum is a vector and its direction is determined from the right hand rule. The magnitude of the angular momentum vector is ***rp***sinϕ.
- Notice that a particle does not have to rotate about O in order to have angular momentum with respect to O.
- Notice that just as Newton's second law may be written in terms of linear momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

  it may also be written in terms of angular momentum:

$$\vec{\Gamma} = \frac{d\vec{\ell}}{dt}$$

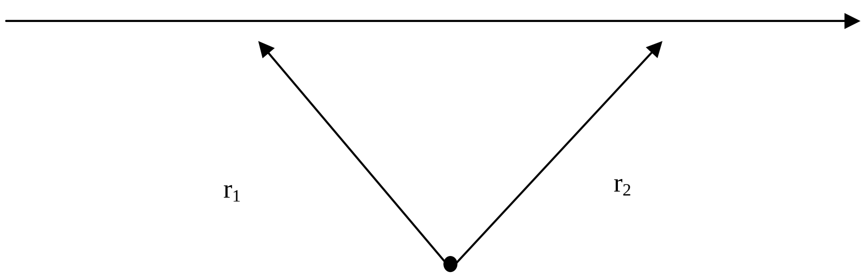
- Angular momentum may be written in terms of moment of inertia and angular velocity for a rigid body and a fixed axis:

$$\vec{\ell} = I\vec{\omega}$$

Angular momentum is an enormously useful quantity in physics for several reasons. First, angular momentum is *conserved*, which means that in the absence of any external torques the angular momentum of a system remains constant.

$$\ell_i = \ell_f$$
$$mvr_i = mvr_f$$
$$I\omega_i = I\omega_f$$

Second, angular momentum may be computed in a wide variety of situations that, at first glance, don't involve rotational motion. All that it really necessary to compute angular momentum is to show motion with respect to any coordinate that one may compute angular momentum with respect to. In the case of instantaneous values this is normally an easy calculation. Complications may arise when computing, for instance, the angular momentum of a particle traveling in a straight line past a fixed coordinate over a short interval (over a long interval *r* changes with the displacement of the
particle).



Notice that the instantaneous magnitude of angular momentum at each of the indicated positions with respect to the origin is the same but that at any other point along the line the angular momentum with respect to the origin will have a different value. How would you calculate the value continuous value of the angular momentum for this system over a large interval?

Translational and Rotational analogs

Newton's second law for translation/rotation:	$\vec{F} = m\vec{a}$	$\vec{\Gamma} = I\vec{\alpha}$
Work in translating/rotating systems:	$W = \int \vec{F} \cdot d\vec{s}$	$W = \int \Gamma d\theta$
Kinetic energy in translating/rotating systems:	$KE = \frac{1}{2}mv^2$	$KE = \frac{1}{2}I\omega^2$
Power in translating/rotating systems:	$P = Fv$	$P = \Gamma\omega$
Momentum in translating/rotating systems:	$\vec{p} = m\vec{v}$	$\vec{\ell} = I\vec{\omega}$

Example 1. A man stands at the center of a turntable holding his arms extended horizontally with a 5 kg mass in each hand. He is set in motion with an angular velocity ω = 5 rev/s. Assume that the moment of inertia of a man is about 6 kg·m^2 and that his arms are 1 meter in length. What is his angular speed if he drops his arms to his sides resulting a final distance of the masses from his center of rotation of 0.2 meters?

Compute the total moment of inertia:

$$I_{total} = I_{man} + I_{weights} = 6kg \cdot m^2 + mr^2$$

$$\rightarrow I_{initial} = 6kg \cdot m^2 + 10kg \cdot m^2 = 16kg \cdot m^2$$

$$\rightarrow I_{final} = 6kg \cdot m^2 + 0.4kg \cdot m^2 = 6.4kg \cdot m^2$$

Now conserve angular momentum:

$$I_i \omega_i = I_f \omega_f \rightarrow \frac{I_i \omega_i}{I_f} = \omega_f \rightarrow \frac{(16kg \cdot m^2)(5rev \cdot s^{-1})}{6.4kg \cdot m^2} = 12.5rev \cdot s^{-1}$$

Example 2. Our sun will eventually collapse from its current size into a much more compact white dwarf star, losing about half of it's mass in the a series of cataclysmic expansions and contractions that will precede this event. The radius of the white dwarf sun will be about 1% of its current value of 7 × 10^5 km. Although the sun has differential rotation a good average value for its rotational period is currently about 30 days. What will its rotational period be once the collapse into a white dwarf is complete? What does this imply about the properties of white dwarf stars in general?

$$\frac{1rev}{30days} = 0.033\,{}^{rev}\!/_{day}$$

Conserve angular momentum:

$$I_i\omega_i = I_f\omega_f \rightarrow \frac{I_i\omega_i}{I_f} = \omega_f$$

$$\frac{2}{5}MR^2\omega_i = \frac{2}{5}\frac{M}{2}\left(\frac{R}{100}\right)^2\omega_f$$

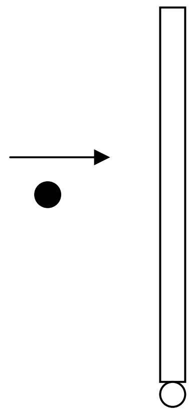
$$R^2\omega_i = \frac{1}{2}\left(\frac{R}{100}\right)^2\omega_f \qquad (\because \text{2/5 and } M \text{ cancel})$$

$$\frac{2R^2\omega_i}{\left(\frac{R}{100}\right)^2} = \omega_f = \frac{(2)\left(7\times10^5\right)^2\left(0.033\,{}^{rev}\!/_{day}\right)}{\left(7\times10^3\right)^2} = 660\,{}^{rev}\!/_{day}$$

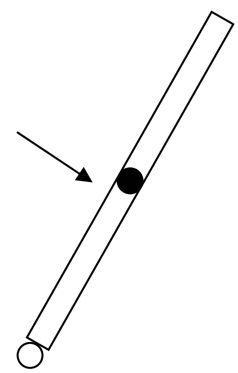
$$\frac{660rev}{day}\times\frac{1day}{1440\min} = 0.46\,{}^{rev}\!/_{\min} \approx 2.2\min$$

Example 3. A bullet, mass = 10 grams, is fired into the center of a door, mass = 15 kg, 1 meter wide with a velocity of 400 m/s. The door is mounted on frictionless hinges. Find the angular speed of the door after the impact.

Consider the type of collision involved here. There is no net external torque exerted on the bullet-door system so angular momentum *is* conserved. The bullet does exert a torque on the door but the door, in return exerts a torque on the bullet so the condition of zero external torques is met.



system before collision



system after collision

Computing angular momentum with respect to the door hinge:

Before: $\ell_{bullet} = mvr = (0.01kg)(400m \cdot s^{-1})(0.5m) = 2.0kg \cdot m^2 \cdot s^{-1}$

$\ell_{door} = 0$

After: $\ell_{system} = I_{system}\omega$

$$I_{door} = \frac{ML^2}{3} = \frac{(15kg)(1.0m)^2}{3} = 5kg \cdot m^2$$

$$I_{bullet} = MR^2 = (0.01kg)(0.5m)^2 = 0.0025kg \cdot m^2$$

$\because$ conservation of angular momentum requires:

$$mvr = I_{system}\omega$$

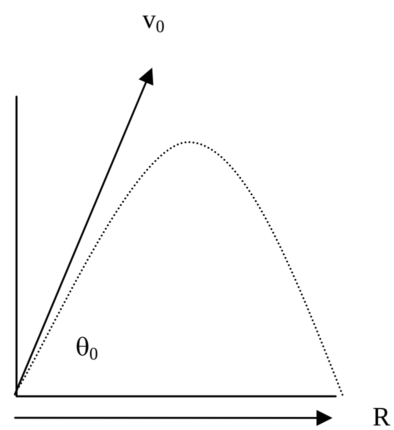
$$2.0kg \cdot m^2 \cdot s^{-1} = (5.0025kg \cdot m^2)\omega$$

$$\omega = 0.4rad \cdot s^{-1}$$

Is energy conserved? $KE_i = \frac{1}{2}m_{sys}v^2 = 800J$, $KE_f = \frac{1}{2}I\omega^2 = 0.4J$

1/2000 of the initial value!

Example 4. A particle of mass *m* is shot with an initial velocity v_0 at an angle of θ_0 as shown below.

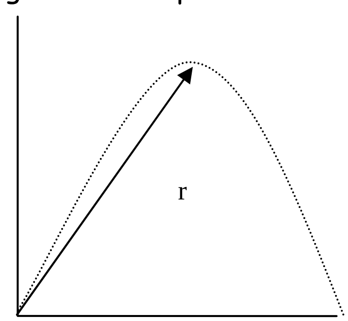


Find the angular momentum of the particle about the origin when the particle is (a) at the origin (b) at the top if its arc (c) just before it hits the ground. What torque causes the angular momentum to change?

(a) When the particle is *at* the origin its angular momentum *about* the origin is zero.

(b) Computing the angular momentum about the origin at the top of the arc.

$$L = \vec{r} \times m\vec{v}$$
$$L = \left(\vec{r}_{\hat{i}} + \vec{r}_{\hat{j}}\right) \times m v_{0xi}$$
$$L = \left(\frac{v_0^2 \sin 2\theta_0}{2g}\hat{i} + \frac{(v_0 \sin\theta_0)^2}{2g}\hat{j}\right) \times m v_0 \cos\theta_0 \hat{i}$$



Taking the cross product:

$$\begin{array}{ccccc}
\hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\
\dfrac{v_0^2 \sin 2\theta_0}{2g} & \dfrac{(v_0 \sin\theta_0)^2}{2g} & 0 & \dfrac{v_0^2 \sin 2\theta_0}{2g} & \dfrac{(v_0 \sin\theta_0)^2}{2g} \\
mv_0 \cos\theta_0 & 0 & 0 & mv_0 \cos\theta_0 & 0
\end{array}$$

$$= -mv_0 \cos\theta_0 \frac{(v_0 \sin\theta_0)^2}{2g} \hat{k}$$

(c) Computing the angular momentum about the origin just before the projectile hits the ground.

$$L = \vec{r}_{\hat{i}} \times m\vec{v}$$

$$L = \frac{{v_0}^2 \sin 2\theta_0}{g} \hat{i} \times m\left(v_0 \cos\theta_0 \hat{i} - v_0 \sin\theta_0 \hat{j}\right)$$

Taking the cross product:

$$\begin{array}{ccccc}
\hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\
\dfrac{v_0^2 \, sin\, 2\theta_0}{g} & 0 & 0 & \dfrac{v_0^2 \, sin\, 2\theta_0}{g} & 0 \\
mv_0 \, cos\, \theta_0 & -v_0 \, sin\, \theta_0 & 0 & mv_0 \, cos\, \theta_0 & -v_0 \, sin\, \theta_0
\end{array}$$

$$= -mv_0 \sin\theta_0 \frac{{v_0}^2 \sin 2\theta_0}{g} \hat{k}$$

Gravity exerts a torque in the –z direction.

Gyroscopes and Tops

Consider Fig 11.19. (From *Physics for Scientists and Engineers*, Serway, 3rd Ed.)

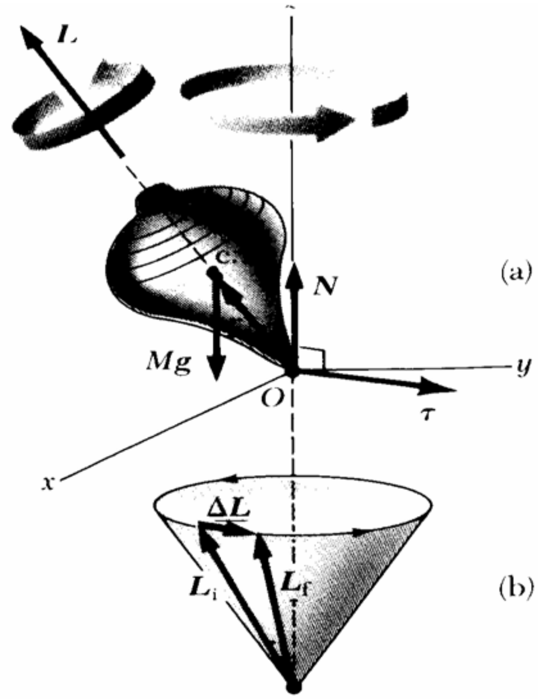


Figure 11.19 Precessional motion of a top spinning about its axis of symmetry. The only external forces acting on the top are the normal force, ***N***, and the force of gravity, ***Mg***. The direction of the angular momentum, ***L***, is along the axis of symmetry.

- When a top is first "spun up" if it is done so with sufficient angular velocity and oriented so that it is upright it stays in that position as it spins.
- By virtue of its spin the top produces an angular momentum vector, *L*, that points along the axis of spin.
- For a while the top, because of its rotational inertia, is stable and is not pulled over by the force of gravity (a condition known as *sleeping*).
- At some point it slows below a certain angular velocity known as the *critical speed* and begins to topple.
- Before the top topples completely over it precesses, or begins a canted orbit around the original spin axis of the system as shown in fig 11.19.
- Since $\Gamma = \frac{d\vec{L}}{dt}$ the torque produced by gravity, which becomes non-zero as soon as the top begins to tip over, produces a *change* in the angular momentum of the system which is in the same direction as Γ (or τ) , i.e., perpendicular to the plane containing ***r***, ***Mg***, and ***L***.
- Note that the magnitude of L does not change (at least not very rapidly), rather it is the direction of ***L*** that changes resulting in "wobble" or precessional motion, or the deflection of the angular momentum vector, ***L***, about the original spin axis of the system.

Angular Momentum as a Fundamental Unit

- Angular momentum plays a role in microscopic as well as macroscopic physics.
- In quantum mechanics the fundamental unit of angular momentum is related to a quantity known as Planck's constant: $h = 6.626 \times 10^{-34}\, J \cdot s$,
- The fundamental unit of angular momentum $\hbar = 1.054 \times 10^{-34} \frac{kg \cdot m^2}{s^2}$, which may be written h/2π.
- Of great importance is the notion that angular momentum is *quantized*, i.e., that all values of angular momentum occur in integer multiples of this fundamental value even for macroscopic systems.
- Note that this fundamental unit has an exceedingly small value. What does this imply about quantization of angular momentum for macroscopic systems?