AC RLC Circuits

AC circuits contain a continuously varying current and voltage that oscillates sinusoidially with time.

- LC circuits produce an oscillating EMF/current.
- Rotation loops produce alternating EMFs/currents.

We can express such an alternating voltage in terms of either a sin or cosine:

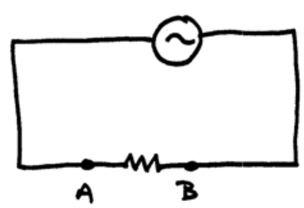
$$v = V\cos\omega t = V\cos\theta$$

v = instantaneous voltage
V = maximum voltage
ω = angular frequency

We'll use *phasor diagrams* to analyze AC circuits.

Resistance, Inductance, Capacitance in AC circuits (RLC)

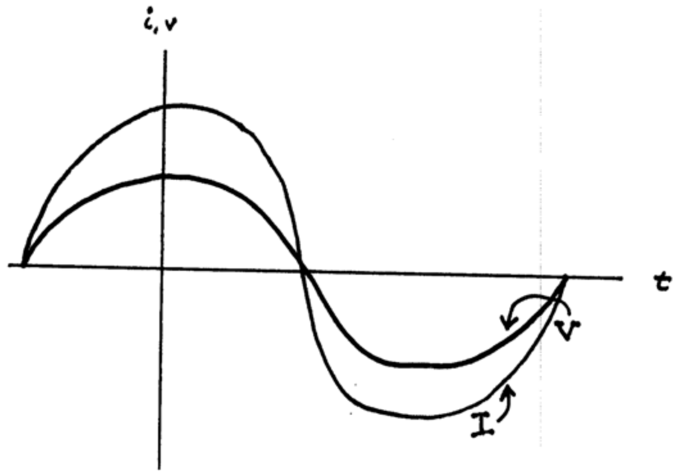
Consider a purely resistive circuit

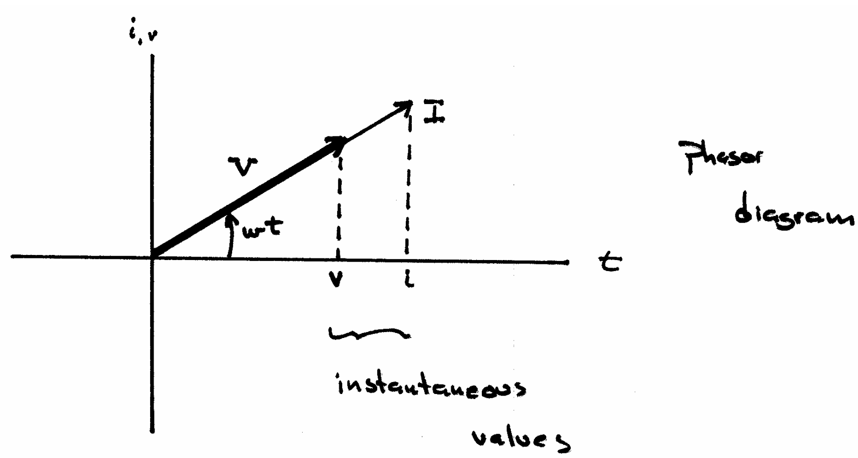


$$v_{ab} = v_R = V\cos\omega t$$

$$i = \frac{v}{R} = \frac{V}{R}\cos\omega t$$

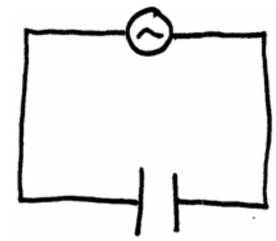
So the current and voltage are both proportional to $\cos\omega t$, i.e., they are *in phase*.





- Phasors are vectors that rotate around the origin of the coordinate system.
- The projections of the phasors of V and I onto the t axis are the instantaneous values of V (v) and I (i) with respect to t.
- Since V and I are in phase their phasors rotate together.
- Notice in the diagrams above that while the values for voltage and current rise and fall together they have different values when plotted on the same set of axes. Why?

Consider a purely capacitive circuit.



The instantaneous charge, *q*, on the capacitor is:

$$q = Cv = CV\cos\omega t$$

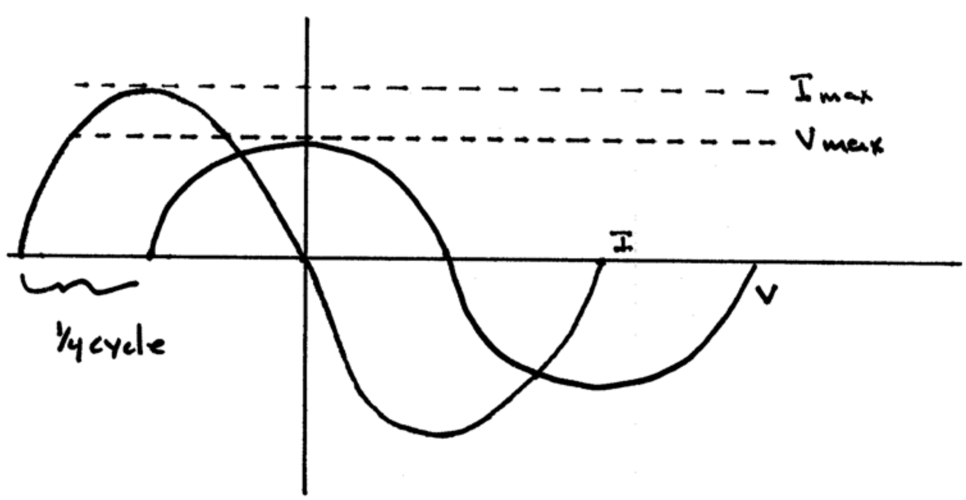
$$i = \frac{dq}{dt} = -\omega CV\sin\omega t$$

Compare: $v_c = V\cos\omega t$

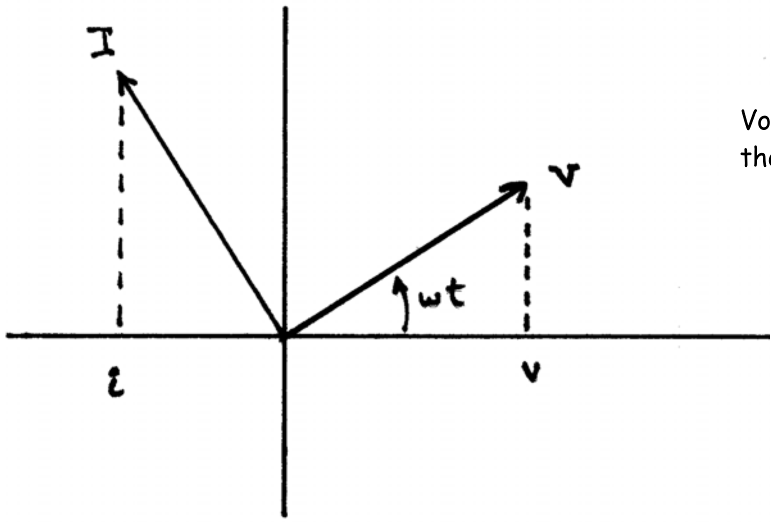
$i_c = -\omega CV\sin\omega t$

$i_c = \omega CV\cos(\omega t + 90^0)$

¼ cycle or 90° phase difference



Notice that the current peaks ahead of the voltage.



Voltage lags the current in the phasor representation

Note: $I_{\max} = \omega CV(1) = \omega CV$

$$I = \frac{V}{R}$$

$$\left.\frac{1}{\omega C} \equiv R \;\rightarrow I = \frac{V}{\frac{1}{\omega C}}\right\}\quad X_c \text{ - capacitive reactance}$$

$$I = \frac{V}{X_c}$$

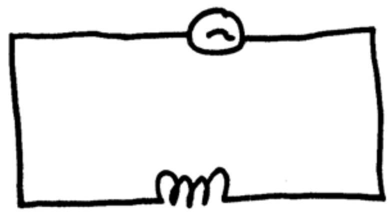
The units of X_c are ohms.

Capacitive reactance acts like resistance in this circuit.

Consider what is happening in a capacitive circuit as the voltage applied to the capacitor increases from zero to some maximum value.

- When there is only a small amount of charge on the capacitor it readily accepts more charge and lots of current flows.
- As the capacitor soaks up charge, the *E* field between its plates increases
- The potential between the plates increases
- The current decreases.
- When V_c reaches its maximum value, the current is zero.

Consider a purely inductive circuit.



Note that the inductor has no DC resistance.

$$v_L = L\frac{di}{dt} = V\cos\omega t$$

$$di = \frac{V}{L}\cos\omega t\ dt$$

$$\int di = \int\frac{V}{L}\cos\omega t\ dt$$

$$i = \frac{V}{L}\frac{1}{\omega}\sin\omega t + C$$

$$i = \frac{V}{\omega L}\sin\omega t + C$$

We use the ic^s to evaluate C

at $t = 0$, $i = 0 \rightarrow C = 0$ so:

$$i = \frac{V}{\omega L}\sin\omega t$$

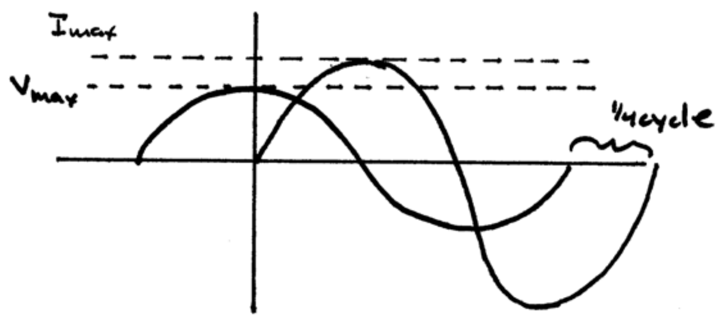
Compare: $v_L = V\cos\omega t$

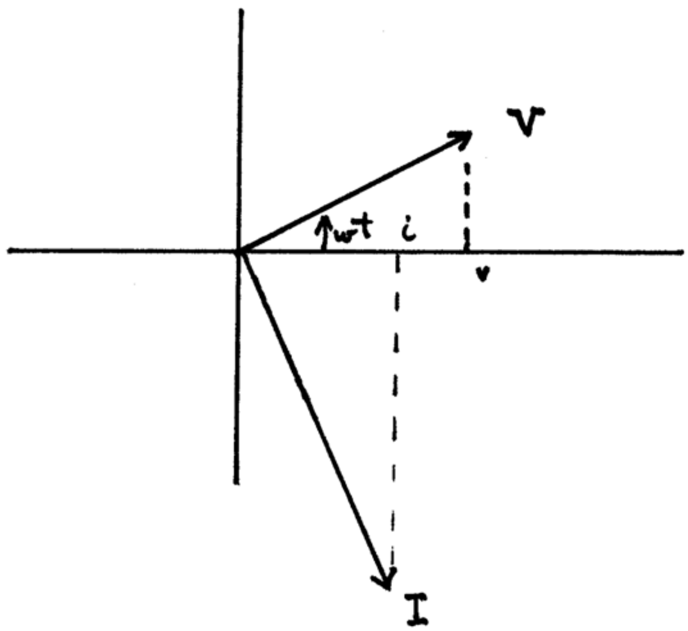
¼ cycle or 90° phase difference

$$i_L = \frac{v}{wL}\sin wt$$

$$i_L = \frac{v}{\omega L}\cos(\omega t - 90°)$$

In this case the current peaks behind the voltage





In an inductor the voltage peaks 90° ahead of the current.

$$I_{\max} = \frac{V}{\omega L}$$

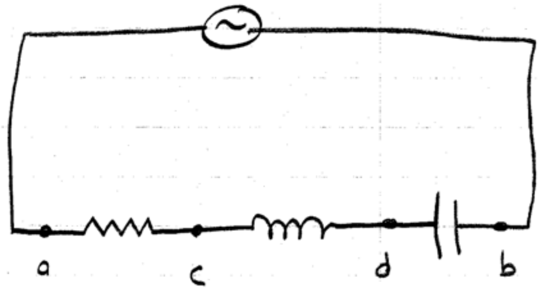
Define $X_L = \omega L$ *inductive reactance* (ohms)

$$I_{\max} = \frac{V}{X_L}$$

- As potential applied to the inductor rises the magnetic flux produces a current that opposes the original current.
- The voltage across the inductor peaks when the current is just beginning to rise, due to this tug of war.

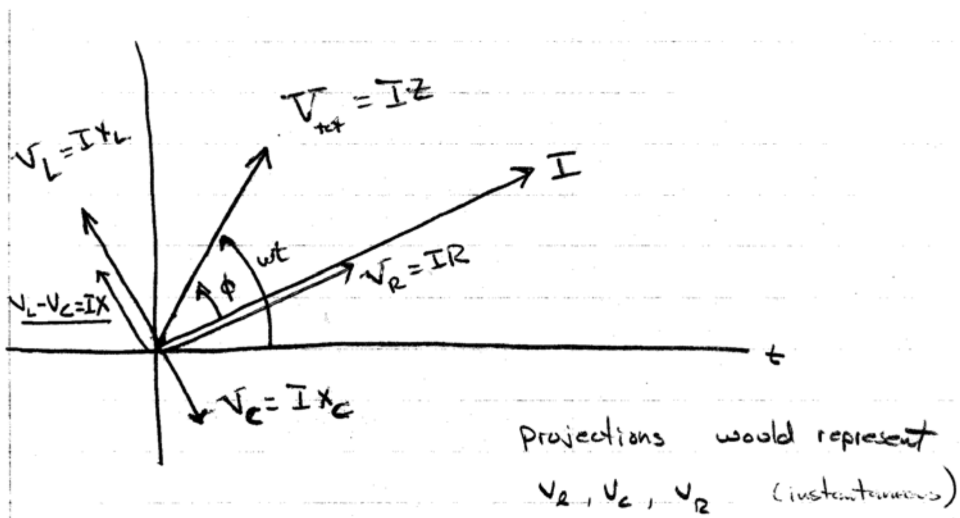
LRC Series AC circuit

Mixture of L, R, C



In analyzing the RLC circuit we'll use phasor diagrams that include phasors for each individual element.

- The current, i , has the same value *everywhere* in the series circuit at the same time, $i = I\cos(\omega t - \phi)$
- V_{tot} – is the voltage across all three components and is equal to the source voltage at that instant ($v = V\cos\omega t$)
- The phasor for V_{tot} is the vector sum of the three individual phasors for the individual voltages.



V_R – voltage across resistor
V_L – voltage across inductor
V_C – voltage across capacitor

maximum values ↓

v_R
v_L
v_C

instantaneous values ↓

- For the resistor *V = IR* and current is always *in phase* with voltage. We use this phasor as the "reference phasor" since it rises and falls with the current.
- For the inductor $V_L = IX_L$ and voltage *leads* the current by 90°
- For the capacitor $V_C = IX_C$ and voltage *lags* the current by 90°
- v_{ab} at any instant is equal to $v_R + v_L + v_C$ (voltage across all three = V_{source})

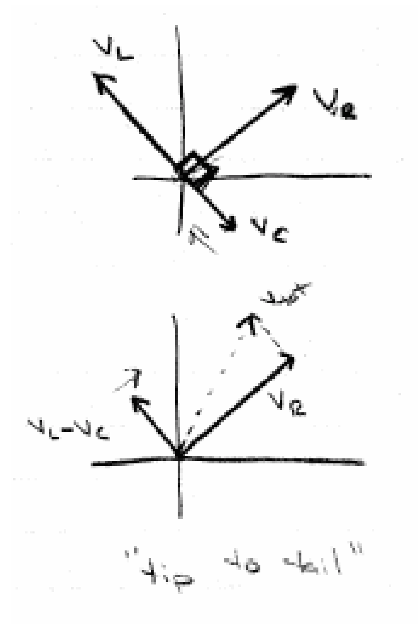
Note: $$V_{ab} = V_R + V_L + V_C$$

$$V = V_R + V_L + V_C$$

$$V_{tot} = \sqrt{V_R^{\,2} + (V_L - V_C)^2}$$

$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2}$$

$$= I\sqrt{R^2 + (X_L - X_C)^2}$$



Define: $\underbrace{X_L - X_C}$

reactance of a circuit

The reactance, *X* of any RLC circuit is: $X = X_L - X_C$

Define impedance, $Z = \sqrt{R^2 + (X_L - X_C)^2}$

$$= \sqrt{R^2 + X^2}$$

So: $$\left.\begin{aligned} V_{tot} &= IZ \\ I &= \frac{V}{Z} \end{aligned}\right\}$$ Note: same form as DC circuits

- The equations relating voltage and current amplitudes have the same form in AC and DC circuits.
- *Z* plays the role of *R* in AC circuits
- *Z* is a function of *R, L, C* and ω.

$$Z = \sqrt{R^2 + X^2}$$

$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

$$Z = \sqrt{R^2 + \left[\omega L - \left(\frac{1}{\omega C}\right)\right]^2}$$

the unit of impedance is the Ohm

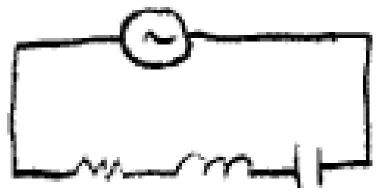
The *phase angle* between the total voltage, *V* and the current *I*, ϕ, is defined (recalling that $X = \left(X_L - X_C\right)$):

$$\tan\phi = \frac{y}{x} = \frac{V_L - V_C}{V_R} = \frac{IX}{IR} = \frac{X}{R}$$

- The phase angle, ϕ, may be positive or negative depending on whether the overall voltage (the sum of the voltage phasors) leads or lags the current in the circuit (which is the same everywhere).
- Instantaneous voltages add algebraically while total voltage amplitudes add vectorially.

Example An series RLC circuit has the following values:

R = 250 Ω
L = 0.6 H
C = 3.5 μF
ω = 377 s^{-1}
V_m = 150 V



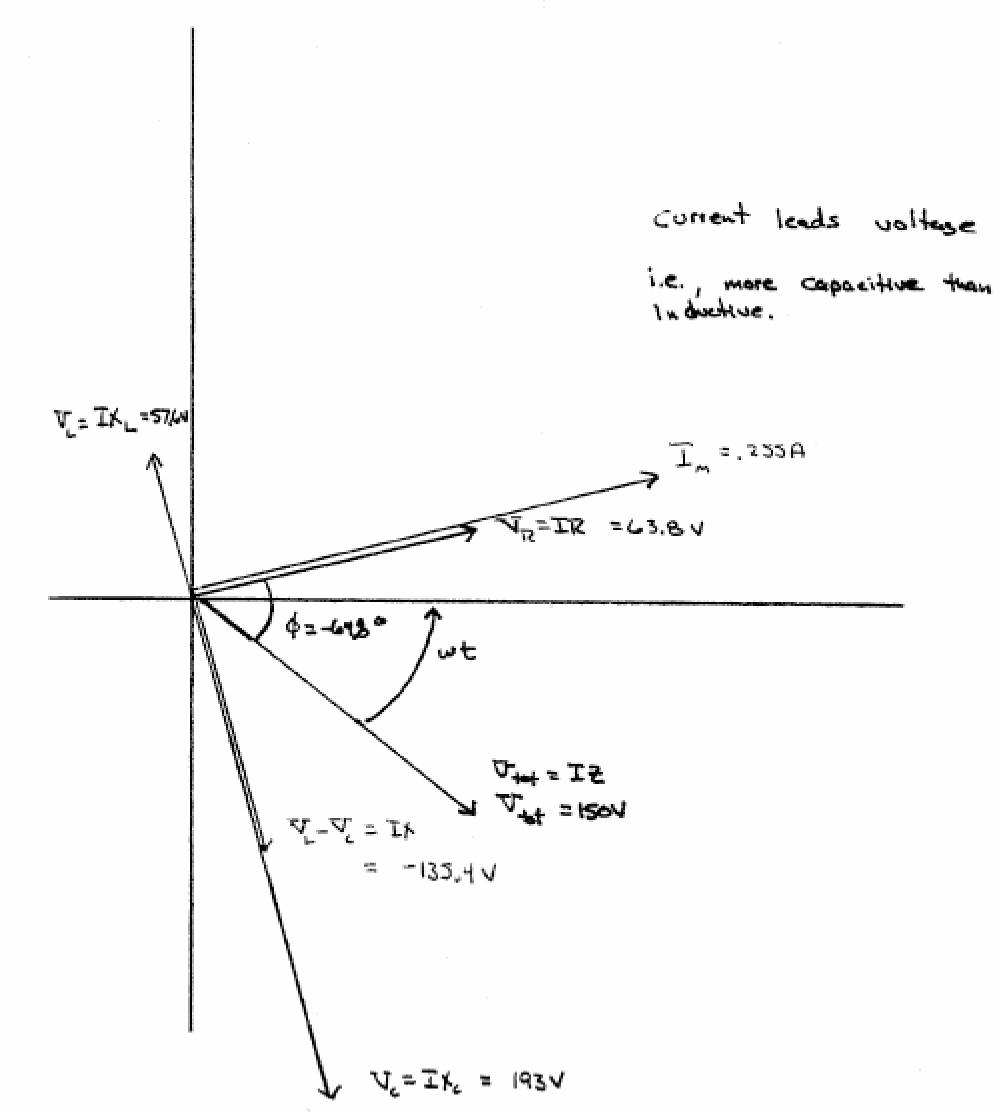
Find a) impedance, b) maximum current, c) phase relationship between the current and voltage (construct a phasor diagram, assume that V_R is in the first quadrant), d) peak voltage across each element, e) instantaneous voltage across each element.

a) $X_L = \omega L = 226\Omega$ $\qquad Z = \sqrt{R^2 + (X_L - X_C)^2} = 588\Omega$

$$X_C = \frac{1}{\omega C} = 758\Omega$$

b) $$I_m = \frac{V_m}{Z} = \frac{150V}{588\Omega} = 0.255A$$

c) $$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = -64.8°$$



d) $V_R = I_m R = 63.8V$
$V_L = I_m X_L = 57.6V$
$V_C = I_m X_C = 193V$

Notice that the sum of these (314V) is greater than V_m (150 V). Peak voltages occur at different times for each element and must be added in a way that takes into account their phase difference.

e) $v_R = V_R \sin \omega t \quad = \quad 63.8v \sin 377t$

$v_L = V_L \cos \omega t \quad = \quad 57.6v \cos 377t$

$v_C = V_C - \cos \omega t \quad = \quad -193v \cos 377t$

Average and RMS Values

- We would like to be able to measure quantities in AC circuits that change with time.
- When one uses a meter to measure the voltage and current in an AC circuit the values are constant (if the meter is on the correct setting) because AC meters read *RMS* values.
- RMS values are weighted averages of time varying (sinusoidal) quantities.

When computing the average of a set of discrete values, one merely adds the terms in the set and divides by the number of terms. Finding the average value of any continuous function is a bit more involved. The average value of a function (f_{av}) that varies with time $f(t)$ from $t_1 \rightarrow t_2$ is:

$$f_{av} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt$$

Consider the time varying AC current: $i = I\sin\omega t$.

The period of this current is: $\tau = \frac{1}{f} = \frac{2\pi}{\omega}$

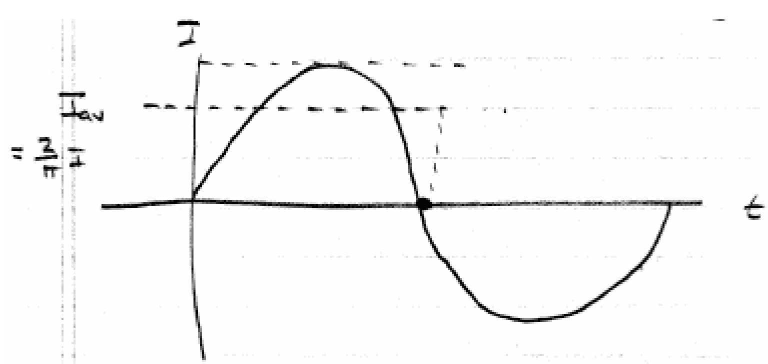
Due to the symmetry of the sine function we may consider just a half cycle when trying to find an average value (the full cycle will give us zero. Why?)

For a half cycle:

$$I_{ave} = \frac{\omega}{\pi}\int_{t=0}^{t=\frac{\pi}{\omega}} I\sin\omega t dt$$

$$I_{ave} = -\frac{\omega}{\pi} I \frac{1}{\omega}(\cos\pi - \cos 0)$$

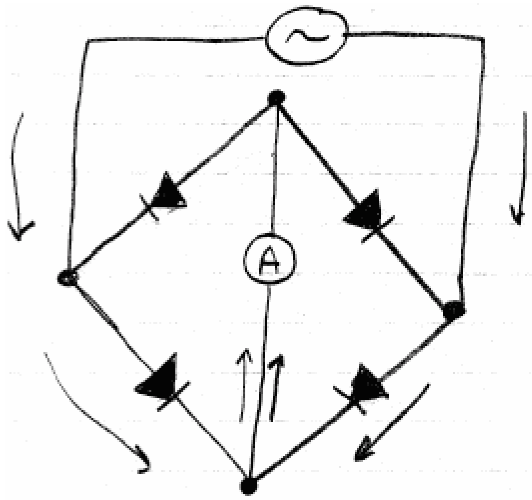
$$I_{ave} = \frac{2I}{\pi}$$

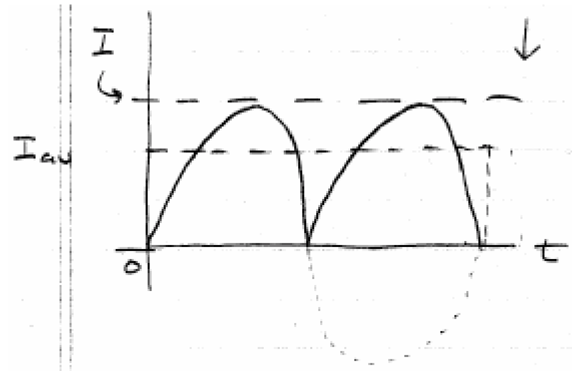


This is about $\frac{2}{3}I$ or about 2/3 the maximum value of *I*.

Notice that for a complete cycle $I_{ave} = 0$. This is true but not very useful.

Consider a bridge rectifier:





Here $I_{av} = \frac{2}{\pi} I$ (not zero) for a complete cycle since the signal now has no negative value4s.

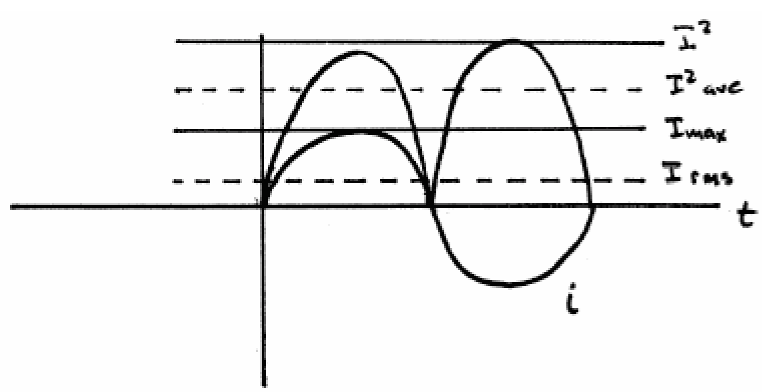
We can set up our meter to give full-scale deflection with a steady current I_0, or when the average value, $\frac{2I}{\pi} = I_o$, or $I = I_0 \frac{\pi}{2}$.

Meters usually read RMS values.

- RMS – root mean square
- Because the square of any quantity is intrinsically positive we avoid the problem of average values of sinusoidal quantities becoming zero over a complete cycle.

$$I_{rms} = \frac{I}{\sqrt{2}}$$

$$V_{rms} = \frac{V}{\sqrt{2}}$$



120VAC is a RMS value :

$V_{RMS} = \frac{V}{\sqrt{2}}$. So peak voltage is about 170VAC for an RMS value of 120VAC

Power in AC circuits

In general power is related to voltage and current as:

$$P = Vi$$

In a resistive circuit:

$$P_{ave} = \frac{1}{2}VI$$

$$P_{ave} = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}} \quad = V_{rms}\ I_{rms}$$

$$P_{ave} = I^2{}_{rms}R$$

Notice that this is the same as for a DC circuit.

In a capacitive circuit: $P_{ave} = 0$ Capacitor cycles through charging and discharging

In an inductive circuit: $P_{ave} = 0$ Inductor cycles fields

Power in LRC combination circuits

Recall that the current and voltage differ by a phase angle ϕ.

$$P = VI$$

$$P = [V\cos(\omega t + \phi)][I\cos\omega t]$$

$$P = \frac{1}{2}VI\cos\phi$$

$$P = V_{rms}I_{rms}\cos\phi$$

$\cos\phi$ - the *power factor* of a circuit

RLC Circuit Summary

In pure resistive circuits: $\phi = 0$ $\cos\phi = 1$ $P = V_{rms} I_{rms}$

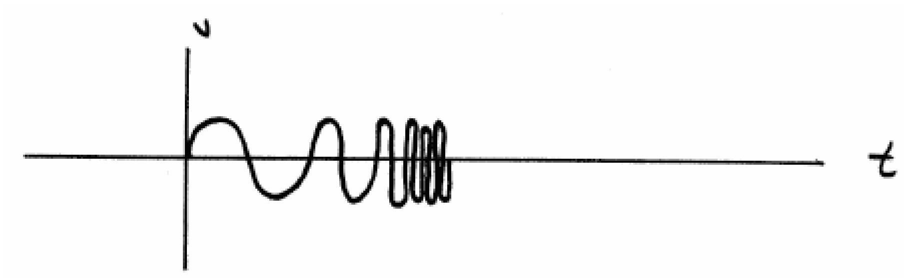
In pure capacitive circuits: $\phi = -90°$ $\cos\phi = 0$ $P = 0$

In pure inductive circuits: $\phi = +90°$ $\cos\phi = 0$ $P = 0$

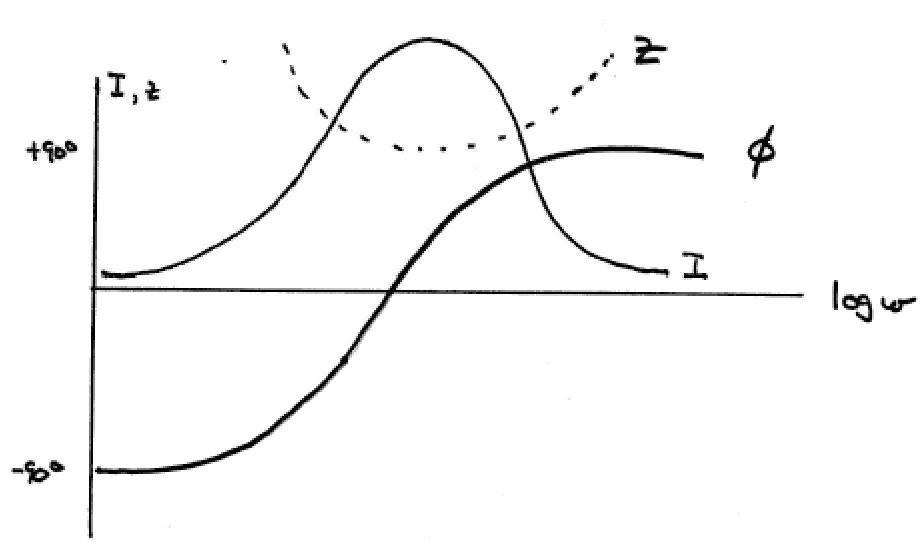
RLC Series Resonance

The impedance of an RLC series circuit varies with frequency.

If one varies the frequency of an AC source while holding voltage constant,



one finds that the current varies with frequency as shown below.



Furthermore it is apparent that:

$$I_{\max} = Z_{\min}$$

- This peaking of current at a specific frequency is known as resonance.
- Electrical resonance is analogous to resonance in mechanical systems.
- As in mechanical systems there is a resonant frequency for any circuit:
  ω_o - resonant frequency

$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

At resonance:

$$X_L = X_c$$

$$Z = R$$

$$\omega_o L = \frac{1}{\omega_o C} \rightarrow \omega_o = \frac{1}{\sqrt{LC}}$$

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Note that $\omega_o = \omega$, is the natural frequency of an LC circuit

$$P_{ave} = I_{rms}^2 R = \frac{V_{rms}^2}{Z^2} R$$

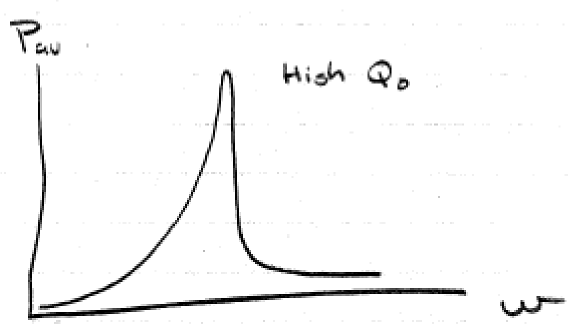
$$P_{ave} = \frac{V_{rms}^2 R}{R^2 + \left(X_L - X_C\right)^2}$$

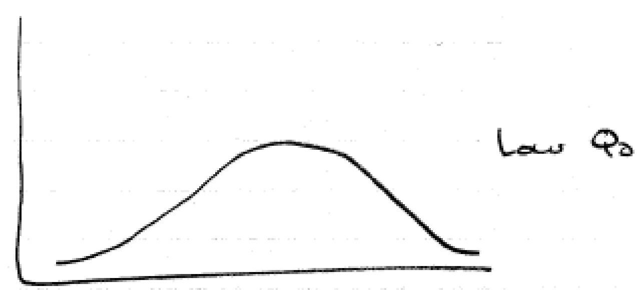
$$P_{av} = \frac{V_{rms}^2 R \omega^2}{R^2\omega^2 + L^2\left(\omega^2 - \omega_o{}^2\right)^2}$$

At resonance $\omega_0 = \omega$, P_{ave} is max and equal to $\frac{V_{rms}^2}{R}$

Define the *quality* of the circuit:

$$Q_0 = \frac{\omega_0 L}{R} \quad \text{(the "sharpness" of a peak)}$$



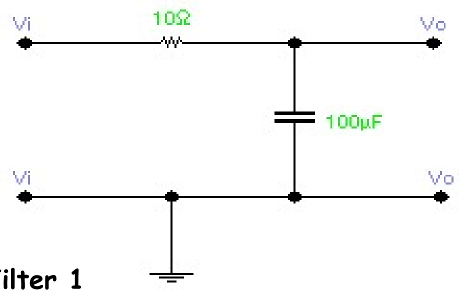


Quality is important in tuning circuits. In practice one can use a capacitor to change frequency and adjust *L* and/or *R* to change sharpness.

Filters

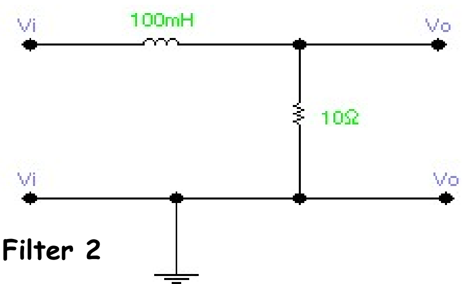
Filters use inductors and capacitors to enhance or diminish limited frequencies in *broad spectrum signals*. Some filters (band pass or shelving filters) will block all frequencies except those desired. Band-pass filters operate as voltage dividers, in that part of the V_i signal goes to ground while the rest passes through the filter.

Consider the low band-pass filter shown below.



Filter 1

For a low band-pass filter, only signals with low frequencies will pass through the filter. For the low band-pass RC filter, shown left, current following the path to ground encounters a capacitor that impedes low frequency alternating signals because the displacement currents, which depend on the rate of change of the electric field in the capacitor, are not large. In this case the signals pass through the filter. When frequencies are high, however, displacement currents are high (because the potential to the capacitor is changing rapidly) and the capacitor provides a ready path to ground.



Filter 2

The RL version of a low band-pass filter works in much the same way except that it's the self-induced emf, which is much greater at higher frequencies, that impedes the flow of current through the inductor with a large back emf. At low frequencies, the inductor doesn't provide much resistance but the resistor does, and the signals pass through the filter.

To quantify the operations of these filter circuits, let's consider each separately, beginning with the low band-pass RC circuit. The input section of the circuit consists of the resistor and capacitor and its impedance is $Z_i = \sqrt{R^2 - X_c^2}$. The output section, which consists of the voltage across the capacitor, is $Z_o = X_c$. The ratio of voltage in and out of the filter is then:

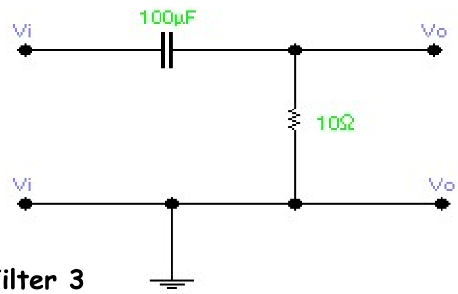
$$\frac{V_o}{V_i} = \frac{Z_o}{Z_i} = \frac{X_C}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \qquad \text{(RC Low Band-Pass)}$$

For the RL low band-pass filter:

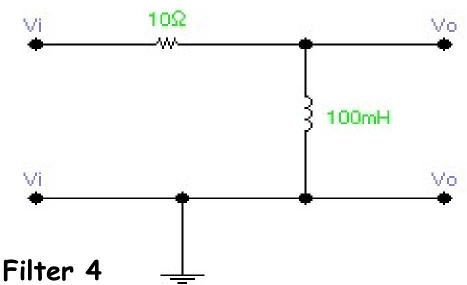
$$\frac{V_o}{V_i} = \frac{Z_o}{Z_i} = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{1 + \left(\omega^2 L^2 / R^2\right)}} \qquad \text{(RL Low Band-Pass)}$$

The *breakpoint frequency*, ω_b, is the frequency at which response between low and high frequencies is $V_o/V_i = 1/\sqrt{2} = 0.707$. For the RC low band-pass filter the breakpoint frequency is $1/RC$ and for the RL low band-pass filter it is R/L.

RC and RL high band-pass filters are essentially the same circuits in reverse.



Filter 3



Filter 4

For the high band-pass RC filter, the rapidly changing field in the capacitor, driven by the rapidly oscillating potential, creates a large displacement current thus allowing the signal to pass through the filter. For the high band-pass RL filter, the inductor creates a large back emf at high frequencies that impedes the path to ground, thus directing the current through the filter.

$$\frac{V_o}{V_i} = \frac{Z_o}{Z_i} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{1}{\sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}} \qquad \text{(RC High Band-Pass)}$$

$$\frac{V_o}{V_i} = \frac{Z_o}{Z_i} = \frac{X_L}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{1 + \left(R^2/\omega^2 L^2\right)}} \qquad \text{(RL High Band-Pass)}$$

For high band-pass filters the ratio of V_o/V_i approaches 1 with increasing frequency. For low band-pass filters this ratio approaches zero. The breakpoint frequencies are the same for both high and low band-pass filters.

Let's consider the values given with the accompanying circuits. Assume that each is driven at 60Hz (377 rad/s). For filter 1, R = 10Ω, C= 100μF so:

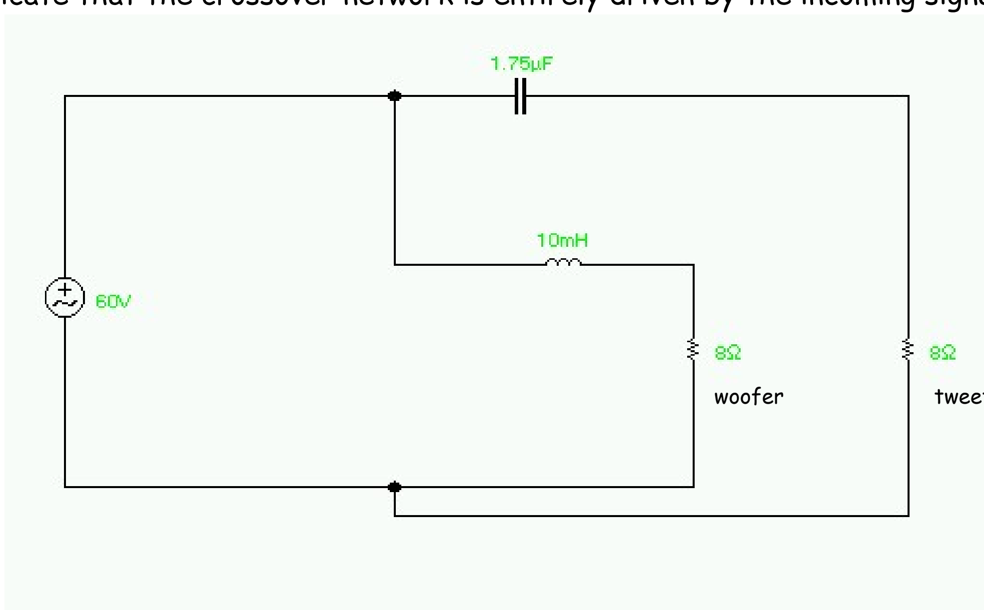
$$\frac{V_o}{V_i} = \frac{Z_o}{Z_i} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} = \frac{1}{\sqrt{1+\left[(377s^{-1})^2(10\Omega)^2(100\times10^{-6})^2\right]}} = 0.94$$

It appears that this circuit is operating near the breakpoint frequency. Let's investigate further:

$$\omega_b = \frac{1}{RC} = 1000s^{-1} = 159Hz$$

Audio Crossovers

Audio crossovers are filters that send audio signals of different frequencies to different processing units. A typical application is a two-way passive crossover network in a loudspeaker enclosure - where low frequencies are sent to a woofer and higher frequencies are sent to a tweeter (most professional boxes consist of three or four-way crossovers powered by different amps). The term passive is used to indicate that the crossover network is entirely driven by the incoming signal.



The circuit above combines an RL low-pass filter with an RC high-pass filter. Frequencies above a certain threshold will be sent to the tweeter and below the same threshold, to the woofer.

Since we want the same response from both circuits, we equate the response equations for the two filters:

$$\frac{R}{\sqrt{R^2 + X_L^2}} = \frac{R}{\sqrt{R^2 + X_C^2}} = \frac{V_o}{V_i}$$

and the responses of both filters is the same when $X_L = X_C$. Thus:

$$\omega L = \frac{1}{\omega C} \rightarrow \omega_{crossover} = \frac{1}{\sqrt{LC}} \rightarrow f_{crossover} = \frac{1}{2\pi\sqrt{LC}}$$

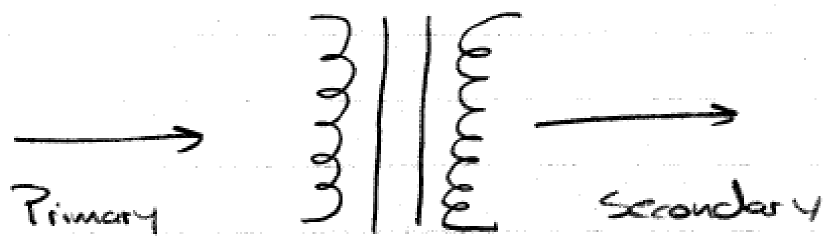
Using the values in the crossover network above, the crossover frequency is found to be approximately 1200 Hz.

Transformers

Advantages of AC over DC

- easier to step up and down
- easier to transmit
- can use high voltage and low current to reduce I^2R losses in transmission lines.

Most transmission lines contain about 500kV that must be stepped down (converted to) lower voltages for household or office operation.



Iron core transformer

Power out always less than power in due to:

- I^2R losses (windings)
- hysteresis (core)
- eddy currents (core)

Usually still better than 90%

We consider only idealized transformers with no losses.

Transformers work by having a different number of turns in the primary and secondary.

In a transformer:

$$\frac{\xi_2}{\xi_1} = \frac{N_2}{N_1} \quad \text{and} \quad \frac{V_2}{V_1} = \frac{N_2}{N_1} \qquad \text{since } V = \xi \text{ (idealized wires)}$$

If $V_2 > V_1$ the transformer is a step up transformer and if $V_2 < V_1$ the transformer is a step down transformer.

It is apparent that:

$$V_1 I_1 = V_2 I_2$$

and:

$$I_1 = \frac{V_1}{\left(\frac{N_1}{N_2}\right)^2 R}$$

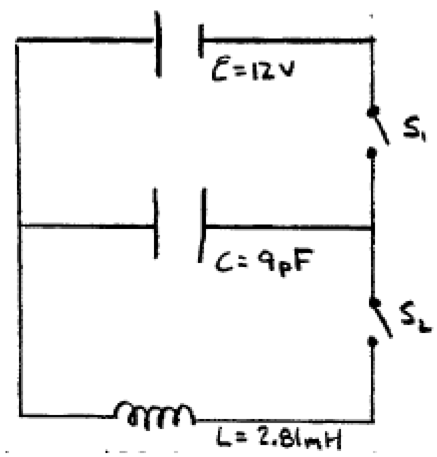
- A transformer transforms current, voltage and resistance (impedance).
- An additional important function of transformers is impedance matching.

Example 1 Design an oscillator circuit with a steady period of about 1 x 10^{-6} seconds. The following parts (not all of which need be used) are available:

- very low resistance wires
- two switches
- a 12 volt battery
- 9 pF, 1.2 μF capacitors
- 2.620 H, 2.814mH inductors
- 100 Ω, 4440 Ω, 50 kΩ resistors

Notice that resistors are not needed since we want an undamped (steady oscillations) circuit.

With $\tau = 10^{-6}\,s$, $f = \frac{1}{\tau} = \frac{1}{2\pi\sqrt{LC}} = 10^{6}\,Hz$ given, L = 2.814mH, C = 9pF satisfy the requirements.



Close S_1 to charge capacitor, then open S_1 and close S_2 to begin oscillations.

What are the maximum values of charge and current in the circuit?

$$Q_m = CV$$
$$Q_m = (9\,pF)(12V)$$
$$Q_m = 1.08\times10^{-10}\,C$$

$$I_m = 2\pi f Q_m$$
$$I_m = (2\pi)(10^{6}\,Hz)(1.08\times10^{-10}\,C)$$
$$I_m = 6.79\times10^{-4}\,A$$

Determine the charge and current as functions of time ($\omega = 2\pi f = 2\pi\times10^{6}\,rad\cdot s^{-1}$).

$$Q = Q_m \cos wt$$
$$Q = 1.08\times10^{-10}\cos\omega t$$

$$I = -I_m \sin\omega t$$
$$I = -6.79\times10^{-4}\sin\omega t$$

What is the total energy stored in the circuit?

$$\frac{Q_m^2}{2C} = \frac{1}{2}LI_m^2 = 6.48\times10^{-10}\,J$$

Example 2

$R = 40\Omega$ $L = 185mH$ $C = 65\mu F$

$V_m = 150v$ $f = 50Hz$ $\omega = 2\pi 50Hz = 100\pi Hz$

$$X_C = \frac{1}{\omega C}$$

$$X_C = \frac{1}{(100\pi)}\left(65 \times 10^{-6} F\right) = 49\Omega$$

$$X_L = \omega L = (100\pi)\left(185 \times 10^{-3} H\right) = 58.1\Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$Z = \sqrt{40^2\Omega + (58.1\Omega - 49\Omega)^2} = 41\Omega$$

$$I_m = \frac{V_m}{Z} = \frac{150V}{41\Omega} = 3.66A$$

a) Across the resistor:

$$V_R = I_m R$$

$$V_R = (3.66A)(40\Omega) = 146V$$

b) Across the inductor:

$$V_L = I_m X_L$$

$V_L = (3.66)(58.1\Omega) = 213V$ (Note: exceeds V_m = 150V)

c) Across the capacitor:

$$V_C = I_m X_C$$

$$V_C = (3.66)(49\Omega) = 179V$$ (exceeds V_m)

d) Across the inductor and capacitor:

$$V_L - V_C = 213v - 179v = 34v$$