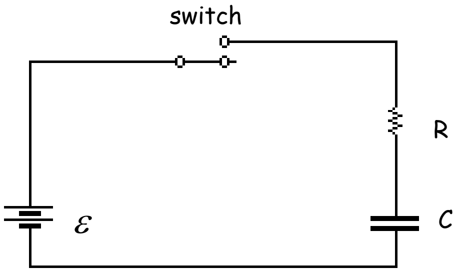
RC Circuits

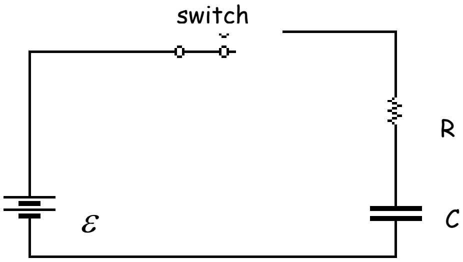
We'll look only at series RC circuits consisting of a capacitor, a resistor and a source of EMF in series

We are concerned with the *transient* behavior of the system, as it approaches *steady state.*

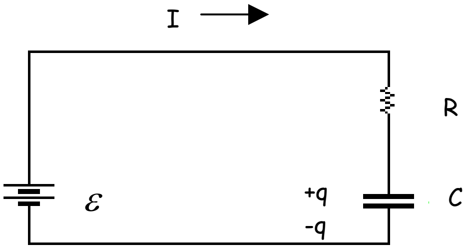


- The maximum charge, *C*, on the capacitor depends upon the applied EMF, $\mathcal{E}$.
- The current in the circuit depends upon time.

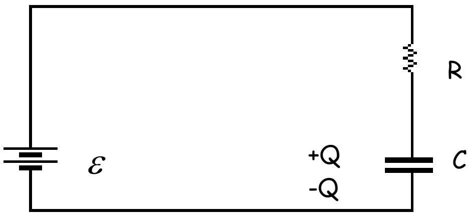
1) $t < 0$ (before the switch is closed) No current flows and the circuit is in a steady state.



2) $t > 0$ (the instant after the switch is closed) Current is at a maximum, the capacitor is charging, the circuit is in a transient state. Potential drop is almost entirely across the resistor.



3) $t \to \infty$ Current approaches zero, capacitor has its maximum charge, and the potential drop is almost entirely across the capacitor. Circuit is in a steady state.



We know the condition of the circuit and all of its components in the initial and final steady states. We seek the condition of the circuit and it's components at any time *t* in between these two states. To do this we'll apply the loop rule to the transient circuit.

Circuit 2: Loop Rule: $\varepsilon - iR - \frac{q}{C} = 0$ Note: *q* and *i* are instantaneous values

In general

- initial conditions: at $t = 0$, $q = 0$, $i = I_0 \Rightarrow \varepsilon = I_o R$ or $\frac{\varepsilon}{R} = I_o$
- final conditions: as $t \to \infty$, $q = Q$, $i \to 0$, $\Rightarrow V = \frac{Q}{C}$ or $CV = Q$
- At the instant the switch is closed current is flowing and the potential drop in the circuit is entirely across the resistor.
- As $t \to \infty$ the current diminishes and the potential drop is entirely across the capacitor.

We seek a time dependent equation for *q* and *I*?

$$\frac{d}{dt}\left(\varepsilon - iR - \frac{q}{C}\right) = 0 \to 0 - R\frac{di}{dt} - \frac{1}{C}\frac{dq}{dt} = 0 \text{ (note: } \frac{dq}{dt} = i\text{)}$$

$$\frac{i}{C} = -R\frac{di}{dt}$$

$$-\frac{1}{RC}dt = \frac{di}{i}$$

$$-\frac{1}{RC}\int_0^t dt = \int_{I_0}^{I} \frac{di}{i}$$

$$-\frac{t}{RC} = \ln\frac{I}{I_0}$$

Recall: $e^{\ln y} = y$

$$e^{-t/RC} = \frac{I}{I_0}$$

$$I(t) = I_0 e^{-t/RC} \text{ or } I(t) = \frac{\varepsilon}{R} e^{-t/RC}$$

Note that this form is indicative of an exponential *decrease* of current with respect to time.

This is a linear, first order homogenous ODE that we solved with separation of variables. Congrats!

Let's look at the behavior of these equations during steady state conditions (when we know a lot about the relationship between *ε, Q, I* and *t*).

- At $t = 0$, $I(t) = I_0$, $\varepsilon = I_o R$ (potential drop is across the resistor)
- As $t \to \infty$, $I(t) \to 0$

Our derivation works at the boundaries of the problem. Particularly at $t = 0$. If we can conjure up a similar relationship for charge on the capacitor as a function of time our derivation will be complete.

$$I = \frac{dq}{dt} \to \frac{dq}{dt} = \frac{\varepsilon}{R} e^{-t/RC}$$

$$dq = \frac{\varepsilon}{R} e^{-t/RC} dt$$

$$\int_0^q dq = \frac{\varepsilon}{R}\int_0^t e^{-t/RC} dt$$

Recall: $\int e^{-ax} dx = -\frac{1}{a} e^{-ax}$

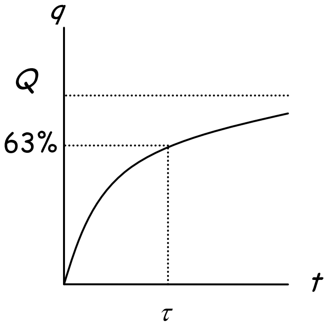
$$q(t) = C\varepsilon\left[1 - e^{-t/RC}\right] = Q\left[1 - e^{-t/RC}\right]$$

Note: the quantity *RC* is known as the time constant for the circuit, τ.

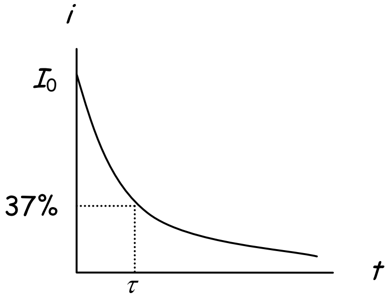
$$q(t) = C\varepsilon\left[1 - e^{-t/\tau}\right] = Q\left[1 - e^{-t/\tau}\right]$$

Note that this expression is indicative of an exponential *increase* of charge with respect to time.

Charging curves of a capacitor



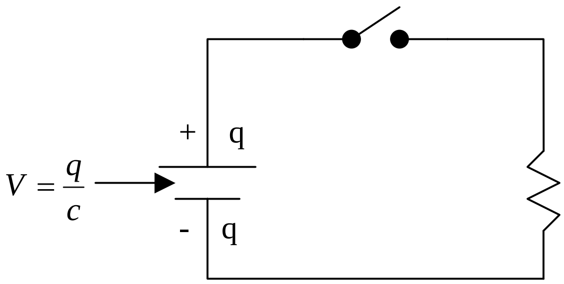
The function $q(t) = Q\left[1 - e^{-t/\tau}\right]$. The time constant represents the time it takes the charge to increase to $\frac{1}{e}$ of its maximum value (*Q*) as the capacitor charges.



The function $I(t) = I_0 e^{-t/RC}$. The time constant represents the time it takes the current to decrease $\frac{1}{e}$ of its original value (I_0) as the capacitor charges.

Note: $RC = \frac{V}{I} \times \frac{Q}{V} = \frac{Q}{I} = \frac{Q}{\frac{Q}{t}} = t$ so τ has units of time

Discharging a capacitor



$V = 0$ since $I = 0$

$IR = \frac{q}{C}$ at $t = 0$

Since charge on the capacitor with decrease as the switch is closed:

$$-R\frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC}dt$$

Using the initial conditions: q = Q at t = 0.

$$\int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}$$

$$q(t) = Qe^{-t/RC}$$

Differentiating this equation with respect to time yields current as a function of time:

$$I(t) = \frac{dq}{dt} = \frac{Q}{RC}e^{-t/RC} = I_o e^{-t/RC}$$

Note that this expression is indicative of exponential decay. Both current and charge decay exponentially.

Note that when *RC* = *t* (one time constant), $V(t) = \frac{V_o}{e}$ = 0.358 V_o or 37% of V_o.

Example If *V* = 10 volts, *R* = 10 ohms, *C* = 1 Farad, find the voltage across the capacitor after 10 seconds, 20seconds.

RC = 10s = τ

$$V_{10} = (10 volts)\, e^{-10/10}$$

$$= \frac{10 volts}{e}$$ = 3.68 volts

$$V_{20} = \quad (10 volts)\, e^{-20/10}$$ = 1.35 volts