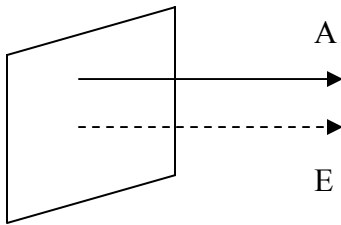


## Electrostatics II

### Gauss's Law

- Gauss's Law relates electric flux to electric charges.
- Electric Flux is the measure of the number of electric field lines penetrating a surface or through some volume.
- If one encloses some net amount of charge within a volume bounded by some surface, the number of E field lines penetrating the surface is proportional to the net charge enclosed.



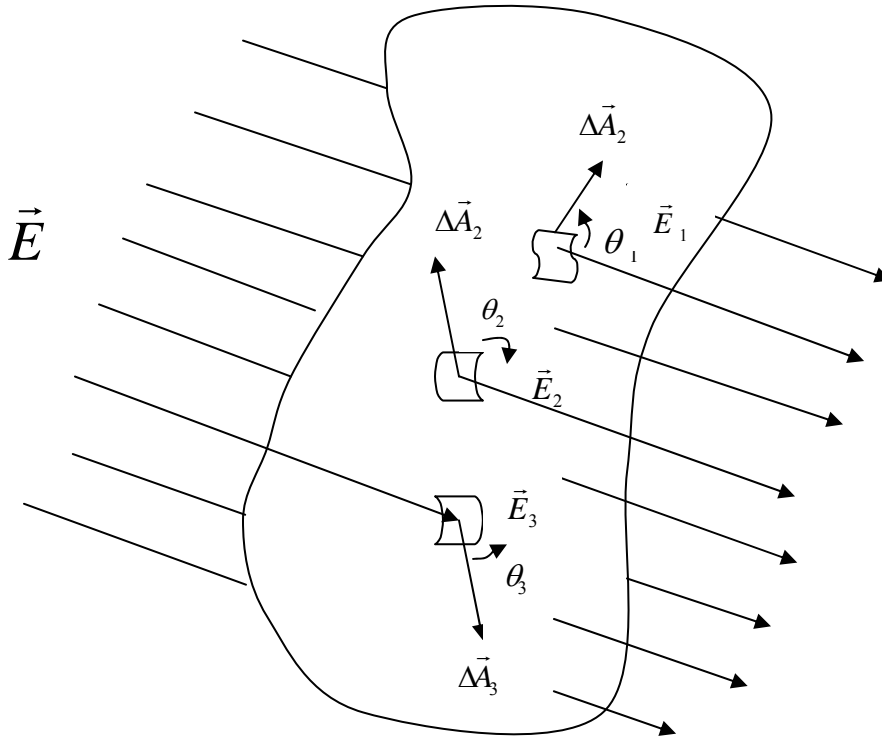
Electric Flux,  $\Phi \equiv \vec{E} \cdot \vec{A} = EA \cos \theta$

Note:  $\Phi_{\max} \equiv EA$  occurs when  $\mathbf{E}$  is parallel to  $\mathbf{A}$ , and when  $\mathbf{E}$  is perpendicular to the surface.

- Maximum flux occurs when  $\mathbf{E}$  is parallel to  $\mathbf{A}$  and minimum flux occurs when  $\mathbf{E}$  is perpendicular to  $\mathbf{A}$
- The number of field lines is independent of the shape of the enclosing surface.
- In general, the E field may not be uniform and may vary over the surface in question, hence:  $\Phi \equiv \int_{\text{surface}} \vec{E} \cdot d\vec{A}$
- Units of electric flux are  $\frac{N \cdot m^2}{C}$

## Flux through an arbitrary volume

Consider an irregular closed surface - a 3D closed surface enclosing some volume penetrated by an external electric field  $\vec{E}$ .



Note:  $\Delta\vec{A}_1, \Delta\vec{A}_2, \Delta\vec{A}_3$ , all point in different directions but all point outward and are normal to the surface.

$$\theta_1 \text{ \& } \theta_2 < 90^\circ$$

$$\therefore \Delta\Phi = \vec{E} \cdot \Delta\vec{A} \equiv + \text{Flux} \quad (\text{out of surface})$$

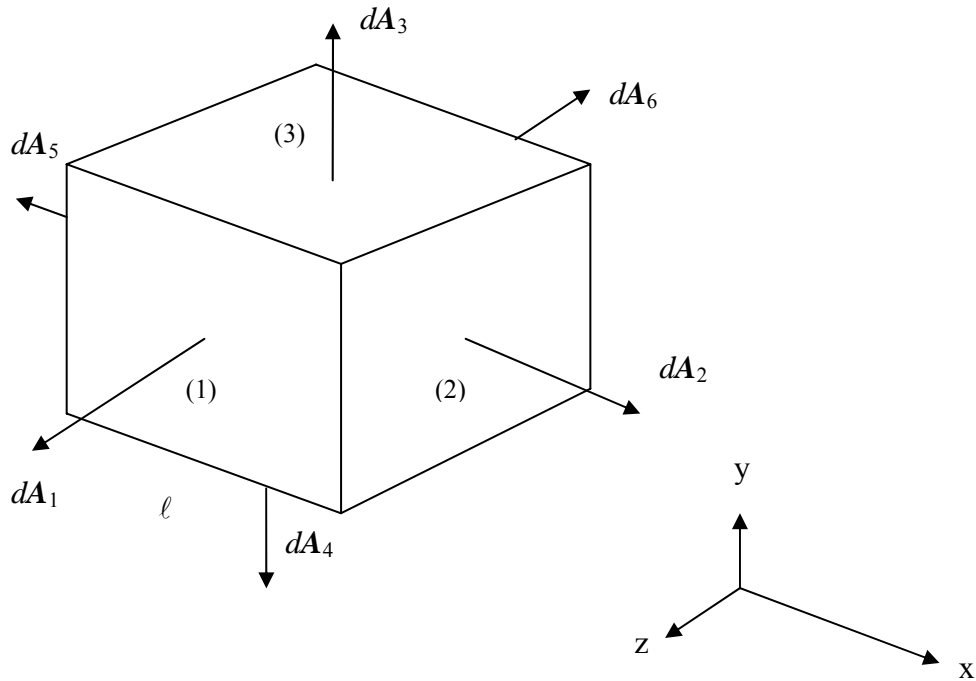
$$\theta_3 > 90^\circ$$

$$\therefore \Delta\Phi = \vec{E} \cdot \Delta\vec{A} \equiv - \text{Flux} \quad (\text{into surface})$$

Net Flux = 0. Flux out = Flux in.

## Flux through a cube

Consider a cube of some volume penetrated by an external field  $\mathbf{E}$ .



$$\Phi_c = \int_5 \vec{E} \cdot d\vec{A} + \int_2 \vec{E} \cdot d\vec{A} \quad (\text{note that the first dot product is negative})$$

$$\begin{aligned}\Phi_c &= -\vec{E} \int_5 d\vec{A} + \vec{E} \int_2 d\vec{A} \\ &= -EA + EA \\ &= -E\ell^2 + E\ell^2 \\ &= 0\end{aligned}$$

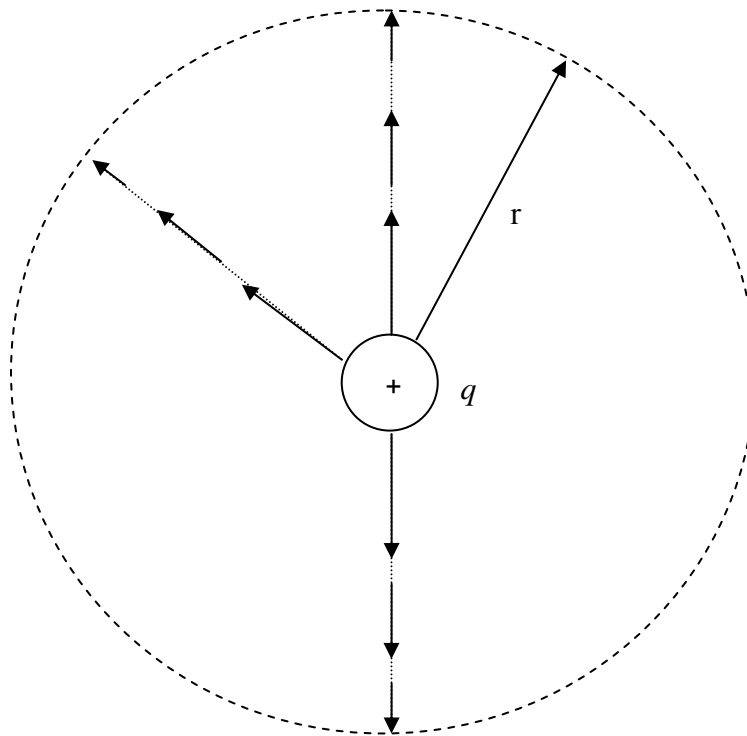
$\vec{E} \cdot d\vec{A}$  for sides 1, 3, 4, 6 = 0 because  $\mathbf{E}$  is parallel to  $\mathbf{A}$  in each case.

The flux in equals the flux out.

## Gauss Law and Electric Field Computation

- Gauss's Law relates  $\mathbf{E}$  with electric charge by relating the electric flux through some artificial bounding surface to the amount of charged enclosed by the surface.
- The enclosing surface is imaginary and is referred to as a *Gaussian surface*.
- Notice that the length of the  $\mathbf{E}$  vectors decreases as the field lines diverge or become less dense. This is how the strength of the field is encoded in the geometry of the field lines.
- A charge outside the surface contributes nothing since it's field lines would go in one side and out the other (net flux = 0)

Consider a point charge  $q$  surrounded by an artificial surface of high symmetry.



Symmetry suggests a spherical Gaussian surface as the best choice for enclosing the charge with an equidistant surface. The choice of such a surface insures:

- On the Gaussian surface  $\vec{E} = \frac{kq}{r^2} \hat{r}$  everywhere (from Coulomb's Law)
- The Gaussian surface is an *equipotential* surface
- All  $\mathbf{E}$  field lines point radially outward and are  $\perp$  to the Gaussian surface.
- $\mathbf{E}$  is everywhere parallel to  $\mathbf{A}$

$$\Phi_c = \oint \vec{E}_n d\vec{a} = \oint \vec{E} \cdot d\vec{a} = E \oint d\vec{a}$$

$$\oint da = A = 4\pi r^2 \quad \text{The surface area of a sphere}$$

$$\Phi_c = EA = E4\pi r^2$$

$$\Phi_c = \frac{kq}{r^2} 4\pi r^2$$

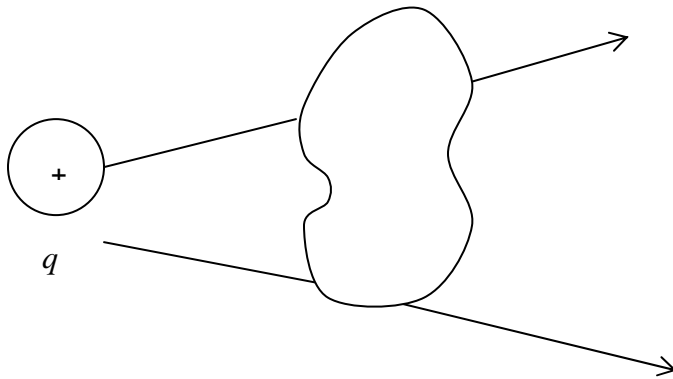
$$\Phi_c = k4\pi q$$

$$\Phi_c = \frac{1}{4\pi\epsilon_0} 4\pi q$$

$$\Phi_c = \frac{q}{\epsilon_0}$$

- Notice that the result is independent of  $r$ .
- The result is proportional to  $\frac{1}{\epsilon_0}$  times the enclosed charge.
- This method of relating flux through a bounding surface to the charge enclosed works for all surfaces of any shape.
- Irregular surfaces are more difficult to integrate but the result is the same.
- In general:  $\Phi_c = \frac{q_{enc}}{\epsilon_0}$
- Gauss's Law works because Coulomb's Law establishes a  $1/r^2$  character for electric force. If this were not true the electric flux through a closed surface would depend on the surface chosen, not just the charge enclosed. Gauss's Law is a consequence of Coulomb's Law.

Consider:



$$\begin{aligned}\Phi_c &= \oint \vec{E} \cdot d\vec{a} \\ &= \frac{q_{enc}}{\epsilon_o} \\ &= 0\end{aligned}$$

Net Flux = Flux in + Flux out = 0

### Gauss Law Summary

$$\Phi_c = \oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_o}$$

Gauss's law finds  $\Phi_c$ . One must do extra work to find  $\vec{E}$ .

Gaussian surfaces must be:

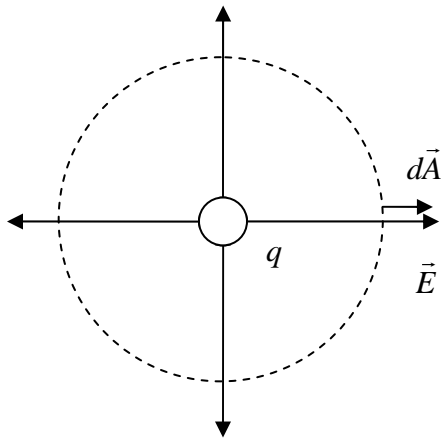
- equidistant from the charge or charge distribution
- equipotential surfaces (necessary from above) meaning that  $E$  is constant over the entire Gaussian surface
- chosen so that  $\vec{E}$  is parallel to  $d\vec{A}$  everywhere
- easy to integrate over to determine the surface area

Gauss' Law is most useful where symmetry lends itself to the constraints above.

Useful symmetries include:

- Spherical
- Cylindrical
- Plane

Computation of the electric field due to a point charge or spherical charge distribution using Gauss' Law.



By choosing the spherical Gaussian surface we guarantee:

- $\vec{E}$  parallel to  $d\vec{A}$
- The Gaussian surface is everywhere equidistant from  $q$ .
- The Gaussian surface is an equipotential surface.
- The Gaussian surface is easy to integrate over (surface area of a sphere)

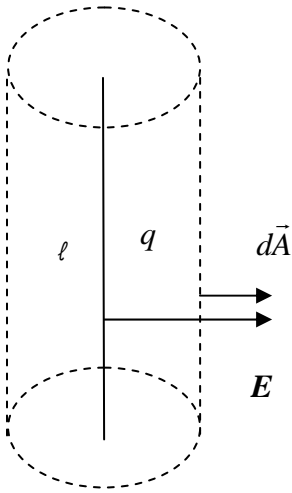
$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$\vec{E} \oint d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$E 4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

$$\vec{E} = \frac{q_{enc}}{4\pi\epsilon_0 r^2} \hat{n} = k \frac{q}{r^2} \hat{n}$$

## Computation of the electric field of a long straight wire using Gauss' Law



By choosing the cylindrical Gaussian surface we guarantee:

- $E$  parallel to  $d\vec{A}$
- The Gaussian surface is everywhere equidistant from  $q$ .
- The Gaussian surface is an equipotential surface.
- The Gaussian surface is easy to integrate over (surface area of a cylinder)

We'll use linear charge density,  $\lambda$ , rather than absolute charge because it suits the symmetry of the problem

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

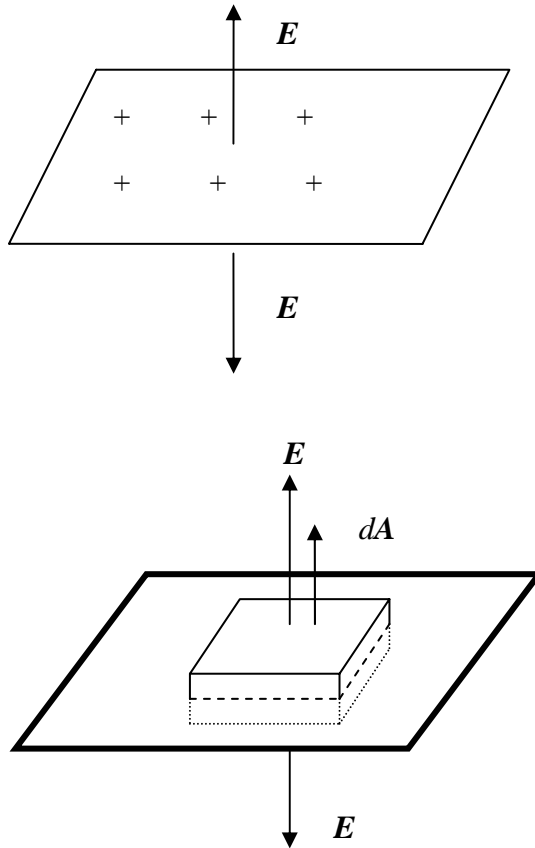
$$\lambda = \frac{q_{enc}}{\ell} \therefore \lambda \ell = q_{enc} \quad \text{using } \lambda \text{ to represent } q$$

$$\vec{E} \oint d\vec{A} = \frac{\lambda \ell}{\epsilon_0}$$

$$E 2\pi r \ell = \frac{\lambda \ell}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{n} = 2k \frac{\lambda}{r} \hat{n}$$

## Infinite Sheet of Charge



A Gaussian "pillbox" seems to suit the sheet symmetry of the charge distribution.

The sides of the pillbox do not contribute since  $\vec{E}$  is  $\perp$  to  $d\vec{A}$  everywhere there.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\vec{E} \oint d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

$$\sigma = \frac{q}{A} \therefore \sigma A = q$$

$$E 2A = \frac{\sigma A}{\epsilon_0}$$

two sides - above and below

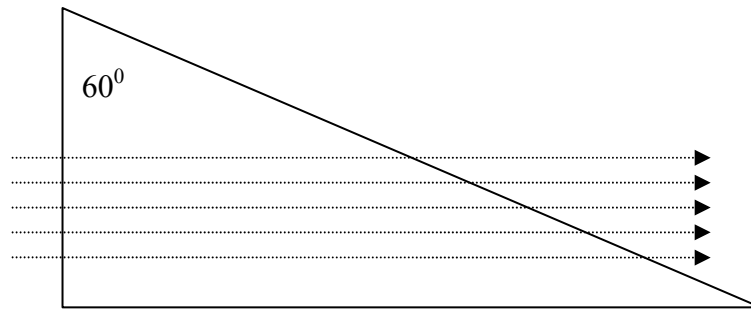
$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

Notice that there is not dependence on  $r$ .

This result could be obtained with a Gaussian surface of different symmetry. Can you think of one?

### Example 1

Consider a closed triangular box as shown below. The magnitude of the uniform  $E$  field directed toward the right is  $7.8 \times 10^4 \text{ N/C}$ . The box is 10cm tall and 30 cm deep. Calculate the electric flux through both the vertical and angled surfaces of the box and the total flux through the box.



- We'll refer to the vertical surface as **A** and the angled surface as **B**
- The area of the surface **A** is:  $10 \times 30\text{cm} = 300\text{cm}^2 = 0.0300\text{m}^2$
- The area of surface **B** is:  $30\text{cm} \times 10\text{cm}/\cos 60^\circ = 600 \text{ cm}^2 = 0.0600\text{m}^2$

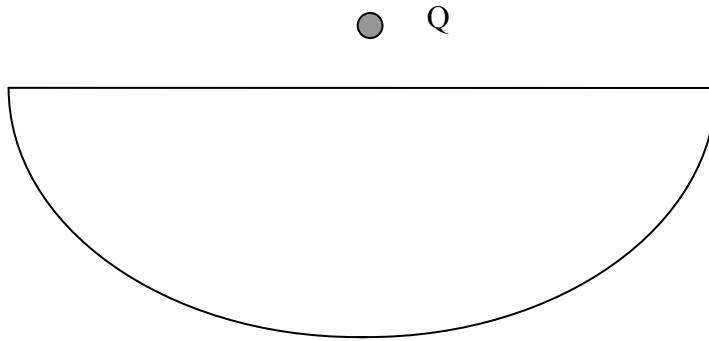
$$\Phi_{E,A} = EA \cos \theta = (7.80 \times 10^4 \text{ N/C})(0.0300\text{m}^2) \cos 180^\circ = -2.34 \times 10^3 \frac{\text{N} \cdot \text{m}^2}{\text{C}}$$

$$\Phi_{E,B} = EB \cos \theta = (7.80 \times 10^4 \text{ N/C})(0.0600\text{m}^2) \cos 60^\circ = +2.34 \times 10^3 \frac{\text{N} \cdot \text{m}^2}{\text{C}}$$

- Since the bottom and other sides of the box are all parallel to the field lines no flux penetrates their surfaces.
- The net flux through the box is zero - flux in = flux out.

### Example 2

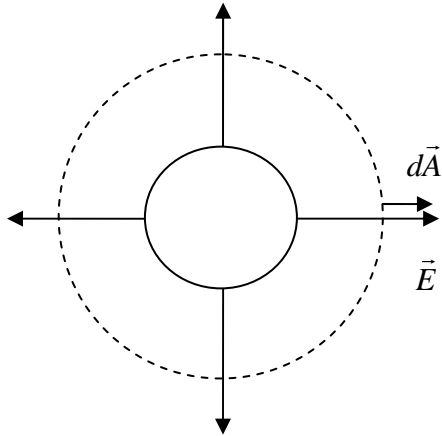
A point charge  $Q$  is placed just above the center of the flat face of a hemisphere of radius  $R$  as shown below. What is the electric flux through both the curved and flat surfaces of the hemisphere?



The flux is *half* that which would be experienced by a full sphere, hence:  $+\frac{Q}{2\epsilon_0}$  for the curved surface and  $-\frac{Q}{2\epsilon_0}$  for the flat surface.

### Example 3

Compute the  $\vec{E}$  field inside and outside of a solid conducting sphere of radius  $R$  containing a charge  $Q$ .



- Since this is a conducting sphere all of the charge is confined to the surface
- No charge exists within the sphere below the surface

**Case I**  $r < R$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \quad \text{Since } q_{enc} = 0, \vec{E} = 0$$

**Case II**  $r > R$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

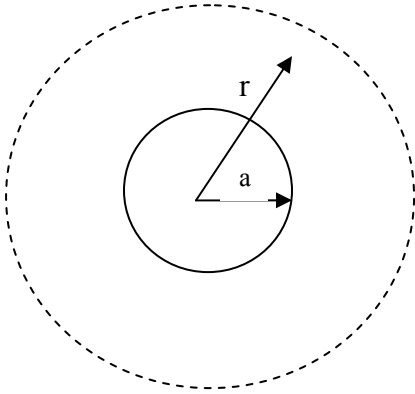
$$\vec{E} \oint d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$E 4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

$$\vec{E} = \frac{q_{enc}}{4\pi\epsilon_0 r^2} \hat{n} = k \frac{Q}{r^2} \hat{n}$$

### Example 4

Consider an insulating sphere of charge of uniform charge density  $\rho$  and total charge  $Q$  (non mobile charge requires a uniform charge distribution). Find  $\vec{E}$  inside and outside this sphere.

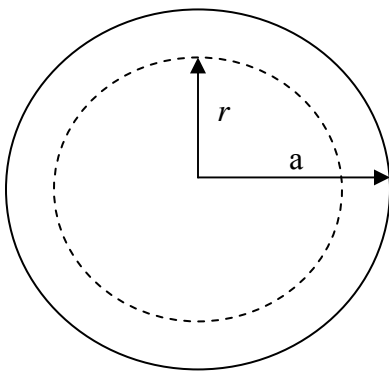


**Case I**  $r > a$

- $\vec{E}$  for  $r > a$  (outside the sphere) is the same as for a point charge or any spherical charge distribution:  $\vec{E} = \frac{kQ}{r^2} \hat{n}$  ( $r > a$ )

**Case II**  $r < a$

- $\vec{E}$  for  $r < a$  is a function of the charge enclosed
- The charge enclosed is a function of radius since uniform charge is distributed throughout the volume of the sphere.



$V'$  = volume enclosed by Gaussian surface

$$q_{enc} = \rho V'$$

$$V' = \frac{4}{3} \pi r^3$$

$$\therefore q_{enc} = \rho V' = \rho \frac{4}{3} \pi r^3$$

$$\oint \vec{E} \cdot d\vec{A} = \vec{E} \oint d\vec{A} = \vec{E} 4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

$$E = \frac{q_{enc}}{4\pi r^2 \epsilon_0} = \frac{\rho \frac{4}{3} \pi r^3}{4\pi r^2 \epsilon_0} \rightarrow \vec{E} = \frac{\rho r}{3\epsilon_0} \hat{n}$$

- Note that as  $r$  gets smaller  $E$  also gets smaller
- What is the value of  $E$  when  $r = 0$ ?
- Note that  $q_{enc}$  for this Gaussian surface is less than the charge enclosed by the sphere itself,  $Q$ .

Let's rewrite the previous expression for  $r < a$ ,  $\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{n}$ , in terms of  $a$  and  $Q$ :

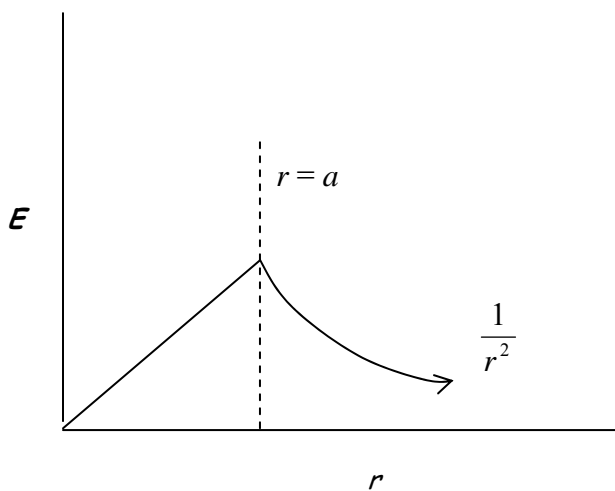
$$\rho = \text{charge/volume} = \frac{Q}{\frac{4}{3}\pi a^3}$$

$$E = \frac{\frac{Q}{\frac{4}{3}\pi a^3} r}{3\epsilon_0}$$

$$\vec{E} = \frac{Qr}{4\pi a^3 \epsilon_0} \hat{n}$$

$$\vec{E} = \frac{kQr}{a^3} \quad r < a$$

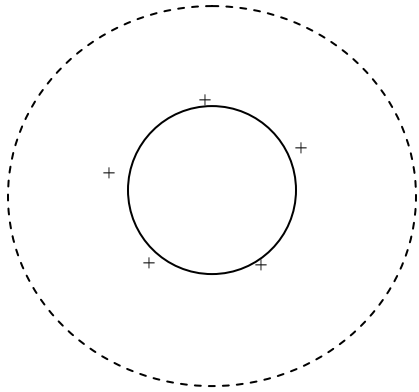
- Note that as  $r$  gets smaller  $E$  also gets smaller
- What is the value of  $E$  when  $r = 0$ ?
- What if this was a conducting sphere?  $E$  inside = 0 because  $q_{enc} = 0$ .



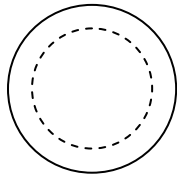
### Example 5

Consider a thin spherical shell with a charge  $Q$  on its surface.

- If the shell is sufficiently thin does it make a difference whether it is an insulator or a conductor?



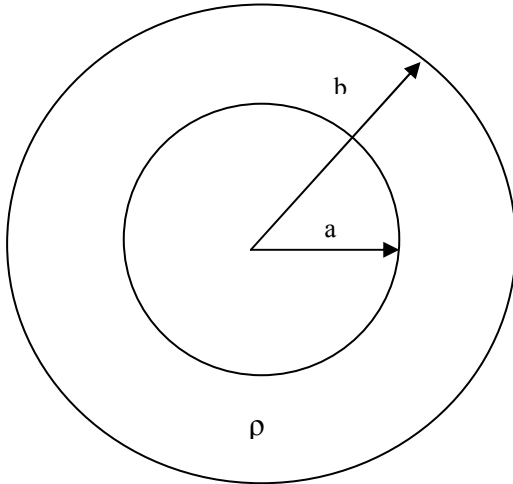
$$\text{Outside: } \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow E4\pi r^2 = \frac{q_{enc}}{\epsilon_0} \rightarrow \vec{E} = \frac{q_{enc}}{4\pi r^2 \epsilon_0} = k \frac{Q}{r^2}$$



$$\text{Inside: } q_{enc} = 0 \rightarrow \vec{E} = 0$$

### Example 6

Consider a hollow insulating sphere. Such a sphere will have charge  $Q$  distributed uniformly throughout the solid spherical shell.



**Case I:**  $r < a$  (inside the hollow portion of the sphere)

$$q_{enc} = 0 \rightarrow \mathbf{E} = 0$$

**Case II:**  $a < r < b$  (within the solid spherical shell)

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow E4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

Evaluating the integral and noting that the charge enclosed in volume  $V'$  is less than the total charge  $Q$ .

$$q_{enc} = \rho V' = \rho \frac{4}{3} \pi (r^3 - a^3)$$

$$E4\pi r^2 = \frac{\rho \frac{4}{3} \pi (r^3 - a^3)}{\epsilon_0} \rightarrow \vec{E} = \frac{\rho 4\pi (r^3 - a^3)}{12\pi r^2 \epsilon_0} = \frac{\rho (r^3 - a^3)}{3r^2 \epsilon_0} \hat{n}$$

**Case III**     $r > b$             (outside the sphere)

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow E4\pi r^2 = \frac{q_{enc}}{\epsilon_0}$$

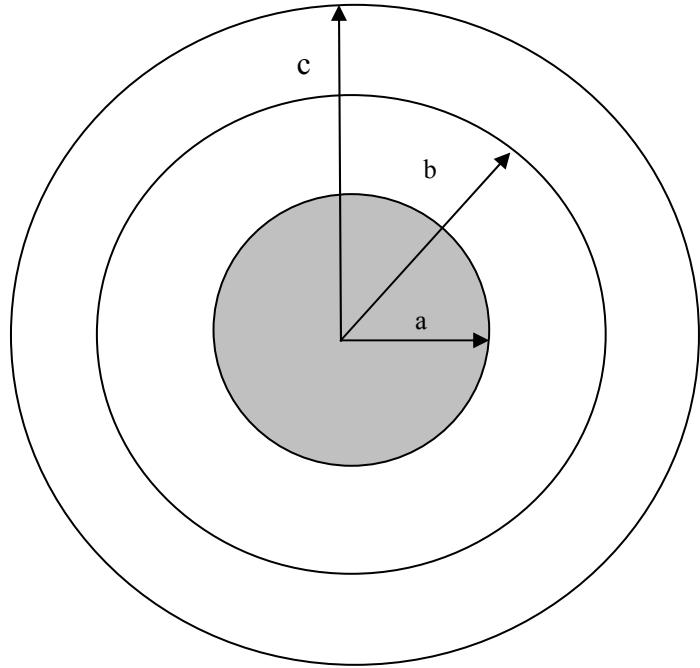
$$E = \frac{q_{enc}}{4\pi\epsilon_0 r^2} = k \frac{Q}{r^2}$$

### **Example 7**

Explain why Gauss's Law may not be used to calculate the electric field near an electric dipole, a triangle with three point charges at its corners or a charged disk.

### Example 8

A solid conducting sphere of radius  $a$  carries a net positive charge of  $2Q$ . A conducting spherical shell of inner radius  $b$  and outer radius  $c$  is concentric with the solid sphere and carries a net charge of  $-Q$ . Find the electric field in the regions:



- 1 -  $r < a$
- 2 -  $a < r < b$
- 3 -  $b < r < c$
- 4 -  $r > c$

1.  $q_{enc} = 0 \rightarrow \mathbf{E}_1 = 0$

2.  $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow E4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = \frac{2Q}{\epsilon_0} \rightarrow E = \frac{2Q}{4\pi\epsilon_0 r^2} = 2 \frac{kQ}{r^2}$

3.  $\mathbf{E} = 0$ . The charge on the inside surface of this conducting shell is  $-2Q$  (by induction), the outer surface carries a charge of  $+Q$  for a total charge of  $-Q$ .

4. The total enclosed charge is  $2Q + (-Q) = Q$  so the  $\mathbf{E}$  field must be:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow E4\pi r^2 = \frac{q_{enc}}{\epsilon_0} \rightarrow \vec{E} = \frac{q_{enc}}{4\pi r^2 \epsilon_0} = k \frac{Q}{r^2}$$