Electricity and Magnetism

A brief history

- Chinese were aware of magnetism 2000 years ago
- Greeks were aware of electricity and magnetism by 700 BCE.
  - Amber, when rubbed with cloth exerted a force on small bits of matter.
  - Magnetite (Fe_3O_3) was attracted to iron.
  - Greek word for amber, *elektron*, is the source of the term *electric*
- Gilbert (1600) distinguished between electric and magnetic forces
- Coulomb (1785) verified that the electric force followed the inverse square law
- Oersted (1820) observed that a compass needle was deflected in the vicinity of an electric wire the same as in the presence of a magnetic field showing that electricity and magnetism were, in fact related.
- Henry/Faraday (1831) separately discovered electromagnetic induction.
- Maxwell (1873) developed his electromagnetic equations which are to e/m what Newton's Laws are to mechanics
- Hertz (1888) verified Maxwell's equations and predictions by generating e/m waves in a laboratory.

Electrostatics I

Electrostatic situations are ones in which charges are stationary, i.e., no *currents* (charges in motion) are present, or are very slow.

In this section we will examine the following electrostatic concepts

- the concept of charge
- charging via induction
- fields and the electric field
- the electric force
- the electric field and test charges
- conductors and insulators

Charge

- has polarity (+ or -).
- is conserved (charge is neither gained nor lost in closed systems – just moved around).
- is quantized (q = N$\pm e$) where $\pm e = 1.602 \times 10^{-19}$ Coulombs (C)
- C – the S.I. unit of charge

The definition of a Coulomb:

With 1 ampere of current (flowing charge) in a wire the amount of charge that flows past a given point in the wire is 1 Coulomb.

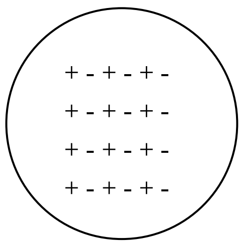
How many electrons/protons in 1 Coulomb of charge?

$$\frac{1C}{1.602\times10^{-19}\ C/p} = 6.25\times10^{18}\ particles$$

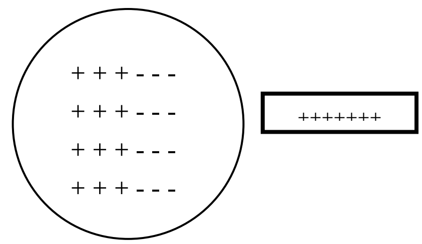
Charging by Induction

Consider a neutrally charged conducting sphere (what is a neutrally charged object?).

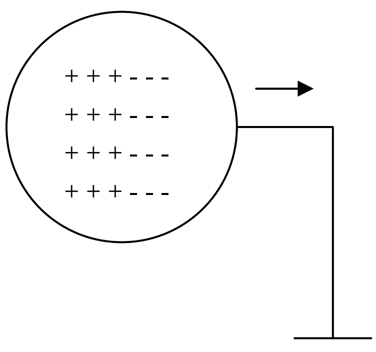
1. Electrically neutral conducting sphere



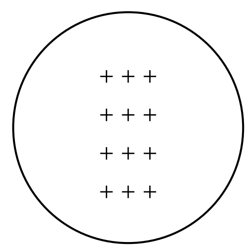
2. The sphere is polarized



3. The sphere is grounded.



4. The sphere is electrically charged



Fields

Fields are the mechanism by which the forces in nature exert influence over a distance. Three types of fields commonly studied in physics are

- Gravity
- Electric
- Magnetic

Although fields exist and are real, our mathematical treatment of them requires the use of an artifice or two.

All fields have the following features

- They exert force along lines of force
- The field *flux* is the number of lines of force per unit area
- They have contours of equal potential energy mapped along equipotential lines
- They have regions of high and low potential energy
- Any field may increase or decrease the kinetic energy of an object placed in the field

In general we wish to compute the field strength and direction for a given situation.

In the case of electric fields we wish to compute both the ***electric force*** that a charge or charge distribution exerts on another charge or charge distribution in the same region of space and/or the ***electric field*** for a given charge or charge configuration at a particular point in space.

Both the electric force and the electric field may be computed with **Coulomb's Law** or **Gauss's Law.**

Coulombs Law is used to compute electric force directly. The electric field may be subsequently determined with relative ease.

Gauss's Law is used to compute the electric field directly, but requires a high amount of symmetry and some greater mathematical sophistication.

The Electric Force

The electric force, $\boldsymbol{F_e}$, is a vector. It is a force just like any other force we've studied.

To directly compute the magnitude and direction of the electric force we use Coulomb's Law.

Although computation of $\boldsymbol{F_e}$ with Coulomb's Law may be tedious for some situations it is generally straightforward.

Coulomb's Law: $\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r}$ $\qquad$ $k \cong 9.0 \times 10^9\ N \cdot m^2 / C^2 = \frac{1}{4\pi\varepsilon_0}$

- ε_0 is the *permittivity* of free space, $8.8542 \times 10^{-12}\ C^2/N{\cdot}m^2$
- *k* (the Coulomb constant) a constant of proportionality
- ***F*** is directed along the line connecting the particles (or charge distributions)
- ***F*** is proportional to the magnitudes of the individual charges (or charge distributions)
- the direction of ***F*** depends upon the polarity of the charges (or charge distributions)
- the electric force is the basis of all contact forces

Compare/contrast the electrical and gravitational forces.

$\vec{F}_e = k\frac{q_1 q_2}{r^2}\hat{r}$	$\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}$
$\frac{1}{r^2}$	$\frac{1}{r^2}$
charged particles	massive particles
attractive/repulsive	attractive

Electric Force computation using Coulomb's Law for point charges

We will consider each point charge, no matter how large in *magnitude*, to be a single point in space, i.e., one with no physical dimensions we have to be concerned with.

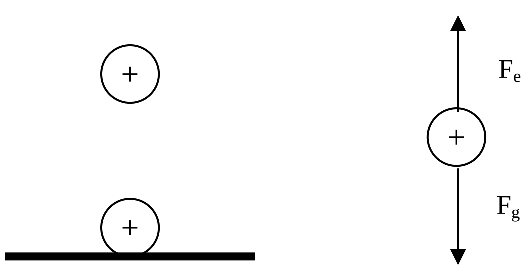
This *does not*, in general, limit each charge to the magnitude of a single charge.

Strategies

- use consistent units (mks)
- be sure to specify the charge being examined in multiple charge configurations
- use absolute values of charges in Coulomb's Law
- establish direction using the polarity of charges
- use the principle of superposition for multiple charges

Example 1

Consider the two point charges below. Ignoring the gravitational attraction *between* the two charges, compute the distance above a proton fixed at the surface of the earth at which a second proton is at equilibrium due to gravitational (between the upper charge and the earth) and electrical forces.

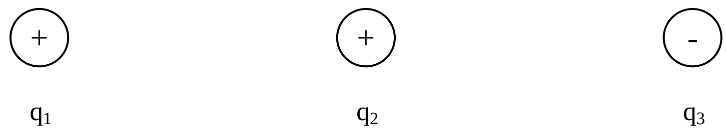


The upper charge is represented in the FBD above. Hence:

$$F_e = F_g \therefore k\frac{|q_1 q_2|}{r^2} = mg \therefore r = \sqrt{\frac{kq^2}{mg}} = 0.118m$$

Example 2

Consider three single point charges along a horizontal line as shown below. Find the direction of the net force on q_1 due to the other two charges.



I'll leave the FBD's for you but they should show:

$\vec{F}_{21} = k\frac{|q_2 q_1|}{r^2} - \hat{r}$ (q_2 exerts a force to the left on q_1)

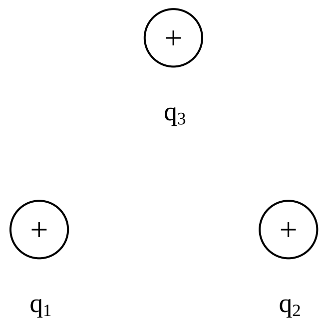
$\vec{F}_{31} = k\frac{|q_3 q_1|}{r^2} + \hat{r}$ (q_3 exerts a force to the right on q_1)

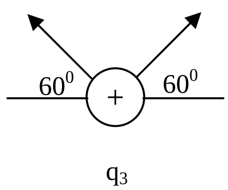
Since $q_2 = q_3$, $F_{21} > F_{31}$ and the net force is to the **left**.

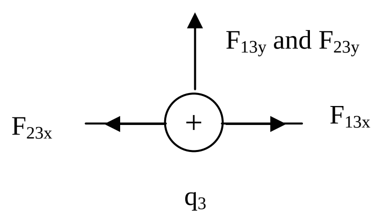
What else would we have to do in order to compute the magnitude of the force on q_1?

Example 3

Consider three single point charges at the corners of an equilateral triangle as shown below. Find the net force on q_3 due to q_1 and q_2.







Note: $F_{13x} = -F_{23x}$ (cancel)
$F_{23y} = F_{13y} = 2F_y$

$$F_{net} = k\left(\frac{|q_1 q_3|}{r^2} + \frac{|q_2 q_3|}{r^2}\right)\sin 60^0$$

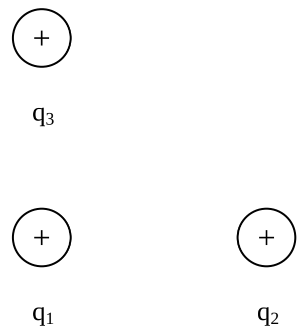
If all three charges have the same magnitude

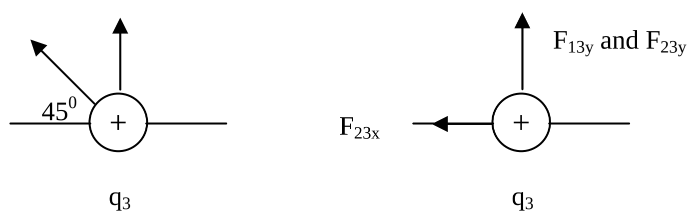
$$F_{net} = 2k\left(\frac{q^2}{r^2}\right)\sin 60^0$$

Make use of symmetry whenever possible in solving problems with Coulomb's Law.

Example 4

Consider three single point charges at the corners of a 45^0 right triangle as shown below. Find the net force on q_3 due to q_1 and q_2.





The symmetry here is less than in the previous problem.

$$F_{net} = F_{13} + F_{23x} + F_{23y}$$

$$F_{net} = k\left(\frac{|q_1 q_3|}{r^2}\right) + k\left(\frac{|q_2 q_3|}{r^2}\right)\cos 45^0 + k\left(\frac{|q_2 q_3|}{r^2}\right)\sin 45^0$$

Coulomb's Law Applied to Charge Distributions

A charge distribution is an extended collection of charges that has physical dimension that cannot be ignored, i.e., anything that is not considered a point charge.

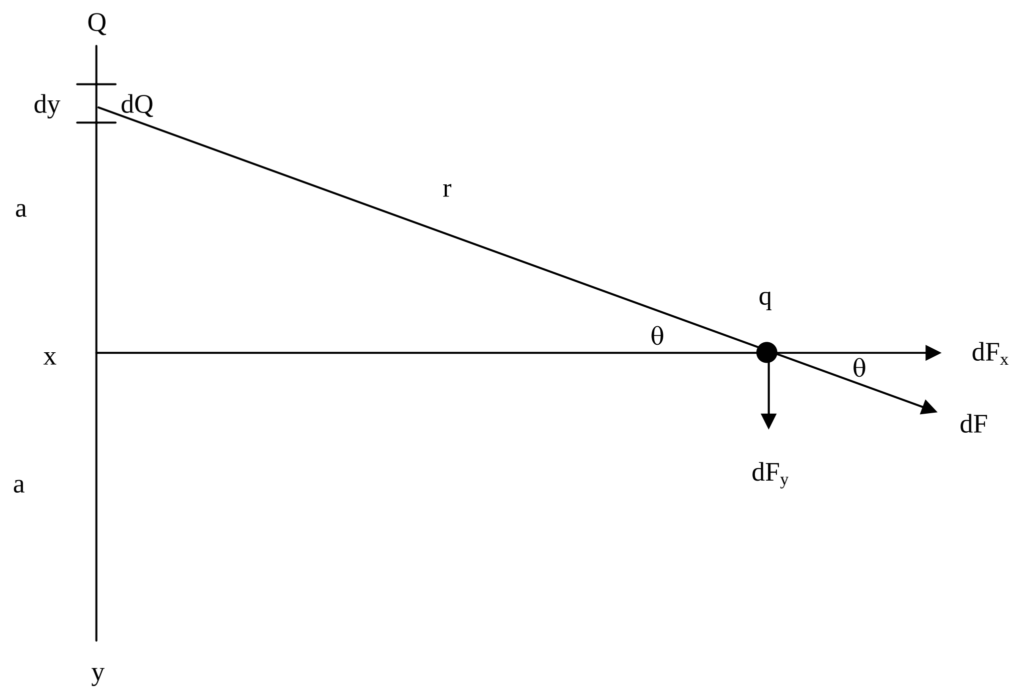
A charge distribution may consists of charge distributed along a line, across a surface or throughout a volume

Coulomb's Law is generally used to compute the force exerted by one charge distribution on another single charge.

In this case the extended charge distribution must be integrated to determine the net force it exerts.

Example 5

Consider a uniform charge distribution *Q*, distributed along a line of length 2*a* from –*a* to +*a*, along the *y* axis, and the force exerted by it on a charge, *q*, some distance *x* from the line.



In order to determine the total force that the line of charge *Q* exerts on the point charge *q* ($\boldsymbol{F}_{Qq}$) we must integrate along *Q* – i.e., add up each differential element of force *dF* that each infinitesimally small chunk of *Q* exerts on *q*.

We begin by noting that each small chunk of charge dQ individually exerts a force on *q* of:

$$dF = k\frac{qdQ}{r^2}$$

In order to get the exact magnitude of each individual element *dF* with the given geometry it will be necessary to express each *dF* in terms of its *x* and *y* components.

In order to set up the needed integral we begin by expressing all quantities in terms that may be conveniently integrated and determining which are constant and which are variable.

a – constant (length of the line of charge does not change as we integrate)
Q – constant (total amount of charge does not change as we integrate)
q – constant (point charge does not change as we integrate)
x – constant (we do not change our position along the x-axis as we integrate)
dQ – differential (the differential chunk of Q)
y – variable of integration (the axis along which we are integrating)
dy – differential (the length of our differential element of charge)
r – variable (the distance from *dQ* to *q* changes as we integrate)

Note: $$\frac{dQ}{Q} = \frac{dy}{2a} \therefore dQ = \frac{Qdy}{2a}$$

For a homogenous linear charge distribution we would expect that the ratio of each differential element of charge (*dQ*) to the total charge (*Q*) should equal the ratio of the *length* of each differential of equal charge (*dy*) to the total length of the charge distribution (2a).

Writing *dF* in terms of *q, Q, dy*, 2a and *r*:

$$dF = k\frac{qQdy}{2ar^2}$$

Noting that *r* may be written in terms of *x* and *y* (eliminating *r* and replacing it with constant *x* and a easily integrated variable, y)

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}} \therefore dF = \frac{1}{4\pi\varepsilon_0}\frac{qQdy}{2a\left(x^2 + y^2\right)} \qquad (1)$$

Notice that each *dF* lies along the hypotenuse of a right triangle with *y* and *x* such that:

$$dF_x = dF\cos\theta, \cos\theta = \frac{x}{r} = \frac{x}{\left(x^2 + y^2\right)^{\frac{1}{2}}} \qquad (2)$$

and

$$dF_y = dF\sin\theta, \sin\theta = \frac{y}{r} = \frac{y}{\left(x^2 + y^2\right)^{\frac{1}{2}}}$$

Notice that by symmetry, all dF_y's cancel, thus:

(2) → (1) $$dF = dF_x = dF\cos\theta = \frac{qQ}{4\pi\varepsilon_0}\frac{xdy}{2a\left(x^2+y^2\right)^{\frac{3}{2}}}$$

The hard part is done! We have our integral to evaluate. The rest is just arithmetic.

$$F = F_x = \frac{qQ}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{xdy}{2a\left(x^2+y^2\right)^{\frac{3}{2}}}$$

(CRC #165) $$\int\frac{dx}{\sqrt{\left(x^2\pm a^2\right)^3}} = \frac{\pm x}{a^2\sqrt{x^2+a^2}}$$

Now - after evaluating the expression above at the limits *a* and –*a* and noting that we are taking a derivative with respect to *y* instead of *x*:

$$\frac{a}{x^2\sqrt{x^2+a^2}} - \frac{-a}{x^2\sqrt{x^2+a^2}} = \frac{2a}{x^2\sqrt{x^2+a^2}}$$

so: $$\left(\frac{x}{2a}\right)\left(\frac{2a}{x^2\sqrt{x^2+a^2}}\right) = \frac{1}{x\sqrt{x^2+a^2}}$$

and: $$\vec{F}_{net} = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{x\sqrt{x^2+a^2}}\hat{i}$$

How can we check this? What if we could compare this result with a simpler geometry that we are more familiar with like two point charges? In that case the force would be simply $\vec{F}_e = k\frac{qQ}{r^2}\hat{r}$.

Note that if *a* is <<<<< *x* the line of charge essentially shrinks to a point as viewed from *q*. In this case:

$$\vec{F}_{net} = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{x^2}\hat{i} = k\frac{qQ}{r^2}\hat{r}$$

Extracting the Electric Field from the Electric Force

We are interested in computing the electric field strength at some point in space some distance from a source charge or charge distribution.

- The electric field strength is proportional to the force the field from some source charge (or charge distribution) exerts on another charge placed at that particular point in space.
- Since the electric field strength is proportional to the magnitude of both the source charge (or charge distribution) *and* the charge at the point in question the strength of the field will vary depending upon the relative strength of *both* sets of charges.
- The addition of the second charge (or charge distribution produces its own field which is superimposed over the source field thus modifying it)
- We would like to be able to measure electric field strength at a point in space that is proportional only to the source charge (or charge distribution) and is independent of everything else.

Test Charges

To satisfy the last condition above we must introduce an artifice known as a *test charge.*

- A test charge has no physical dimension (it is a point charge)
- A test charge is infinitesimally small (violates quantization of charge)
- A test charge is always positive and is denoted with the symbol q'
- A test charge, being infinitely small, produces an infinitely small electric field of its own and exerts an infinitely small electric force on other charges or charge distributions.
- A test charge, when introduced into a region of space near an existing charge or charge distribution, provides a method of interrogating the *in situ* electric field without affecting it.

Now consider the force between a source charge, Q, and a test charge q'

$$\vec{F}_e = k\frac{Q_{source}q'}{r^2}\hat{r}$$

Since the magnitude of q' is infinitely small it may be divided out of both sides of the equation without consequence and the result is the electric field strength at the point in space occupied by q':

$$\frac{\vec{F}_e}{q'} = k\frac{Q_{source}q'}{q'r^2}\hat{r} = k\frac{Q_{source}}{r^2} = \vec{E} \text{ or } \vec{E} = \frac{\vec{F}}{q'}$$

It should be noted that the same result is valid for any real charge q since dividing it out of Coulomb's Law effectively removes it from field computations, hence:

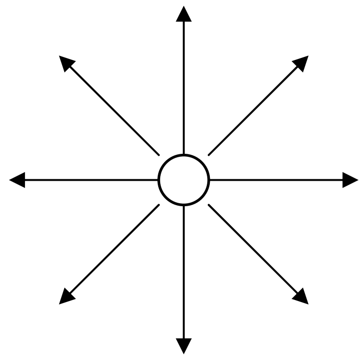
$$\vec{E} = \frac{\vec{F}}{q}.$$

Test charges are used to interrogate regions of space containing source fields without affecting the source field.

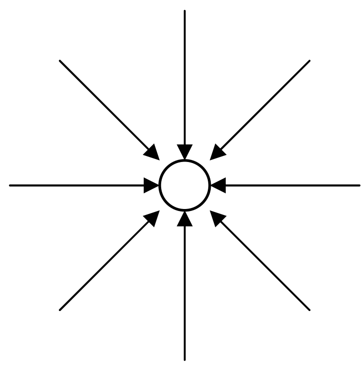
To extract the source electric field strength from the electric force at a point in space occupied by a secondary charge it is all one must do is divide the magnitude of the secondary charge from Coulomb's Law.

Determining the Orientation of an Electric Field Everywhere Surrounding a Charge

Consider the electric field of a single positive point charge, +q.



Contrast that with the electric field of a single negative point charge, -q.

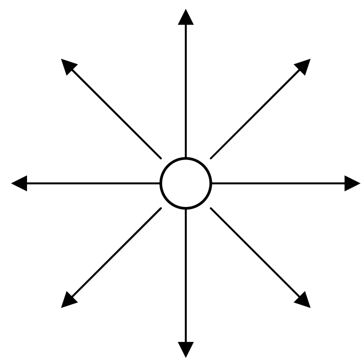


Why the difference in the direction of the arrows?

- Recall that the electric field is an indication of the force a source charge exerts on another charge placed at a particular point in space.
- In order to avoid the difficulties of superimposing fields from both charges we introduce a test charge, q', to determine the effect only of the source charge at the region of space being investigated.
- The test charge, q', is always positive
- Since the first charge is positive, q' would experience a repulsive force radially away from the source charge at any point in space surrounding the source charge.
- Since the second charge is negative, q' would experience an attractive force radially inward.
- In general electric field lines of force may be plotted by determining the direction that a test charge would move it released in the field

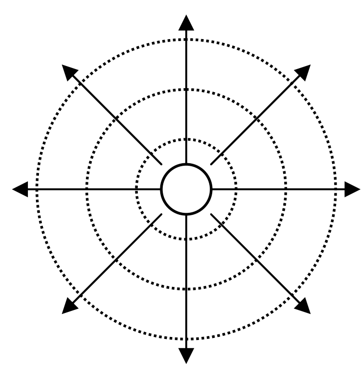
Electric Field Lines of Force and Equipotential Lines

Consider, again, the following charge and its associated electric field



Since the electric field lines are the force vectors for a positive test charge inserted anywhere into the field they are known as *lines of force*.

- The lines of force indicate the direction that a test charge, +q', would move if placed in the field.
- Since ***F*** = m***a***, a test charge let loose in an electric field accelerates
- Since the test charge accelerates it must gain kinetic energy. Work is done.
- If the electric force is conservative, the electric field is conservative and any gain in kinetic energy must be accompanied by a loss in potential energy.
- The potential energy of a test charge is a function of position in the field.



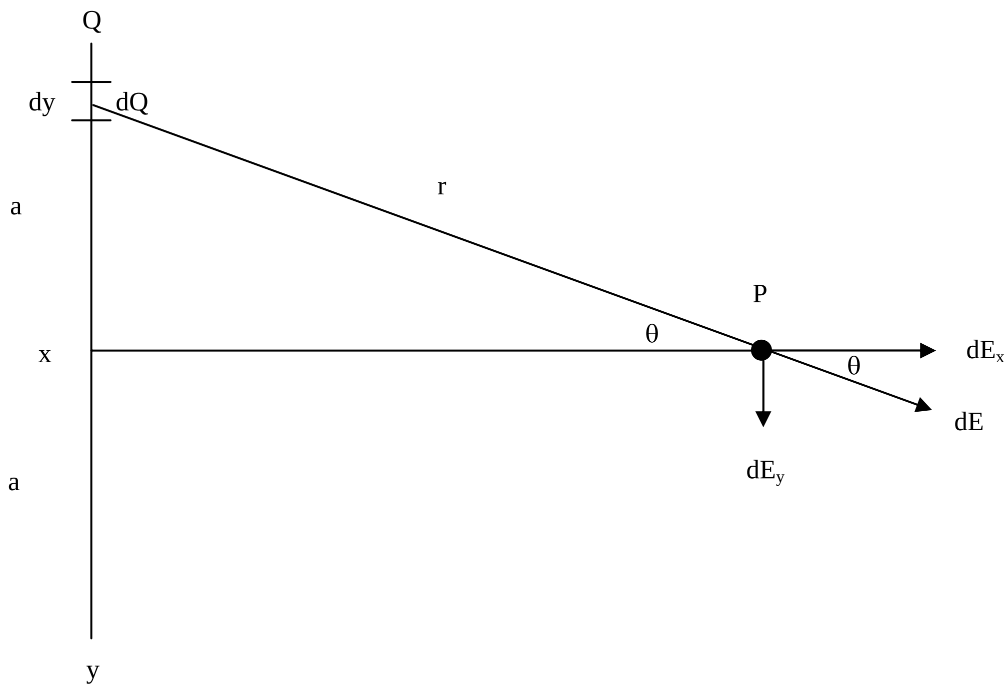
The dashed lines are known as *equipotential lines* and represent contours of equal potential energy within the field.

- What must, at all times, the geometric relationship between equipotential lines and lines of force be?
- Can two lines of force ever cross? What about two equipotential lines?
- Notice that the lines of force represent lines along which maximum work is done in moving a charge in the field. What about the equipotential lines?

We'll revisit the structure of electric fields later but for the time being let's return to the calculation of electric fields at a defined point in space.

Example 6

Consider a uniform charge distribution *Q*, distributed along a line of length 2*a* from –*a* to +*a*, along the *y* axis. Use Coulomb's Law to compute the electric field at *P* due to *Q*.



Initially we proceed as before

$\boldsymbol{E_p}$ = ? $$dE = k\frac{dQ}{r^2}$$

- *a* – constant
- *Q* – constant
- *q* – constant
- *x* – constant
- *dQ* – differential
- *y* – variable of integration
- *dy* – differential
- *r* - variable

Note: $$\frac{dQ}{Q}=\frac{dy}{2a}\therefore dQ=\frac{Qdy}{2a}$$

$$dE=k\frac{Qdy}{2ar^2},r=\left(x^2+y^2\right)^{\frac{1}{2}}\therefore dE=\frac{1}{4\pi\varepsilon_0}\frac{Qdy}{2a\left(x^2+y^2\right)} \qquad (1)$$

$$dE_x=dE\cos\theta,\cos\theta=\frac{x}{r}=\frac{x}{\left(x^2+y^2\right)^{\frac{1}{2}}} \qquad (2)$$

$$dE_y=dE\sin\theta,\sin\theta=\frac{y}{r}=\frac{y}{\left(x^2+y^2\right)^{\frac{1}{2}}}$$

by symmetry, all *dE*$_y$'s cancel

(2) → (1) $$dE=dE_x=dE\cos\theta=\frac{Q}{4\pi\varepsilon_0}\frac{xdy}{2a\left(x^2+y^2\right)^{\frac{3}{2}}}$$

$$E=E_x=\frac{Q}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{xdy}{2a\left(x^2+y^2\right)^{\frac{3}{2}}} \qquad \text{(CRC \#165)}$$

$$\vec{E}_{net}=\frac{Q}{4\pi\varepsilon_0}\frac{1}{x\sqrt{x^2+a^2}}\hat{i}$$

What if *a* is <<<<<*x*? The line shrinks to a point.

$$\vec{E}_{net}=\frac{Q}{4\pi\varepsilon_0}\frac{1}{x^2}\hat{i}=k\frac{Q}{r^2}\hat{r}$$

Define linear charge density $\lambda=\frac{Q}{length}=\frac{Q}{2a}\therefore Q=\lambda 2a$

$$\vec{E}_{net}=\frac{\lambda 2a}{4\pi\varepsilon_0}\frac{1}{x\sqrt{x^2+a^2}}\hat{i}=\frac{\lambda a}{2\pi\varepsilon_0}\frac{1}{x\sqrt{x^2+a^2}}\hat{i}=\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{x\sqrt{\frac{x^2}{a^2}+1}}\hat{i}$$

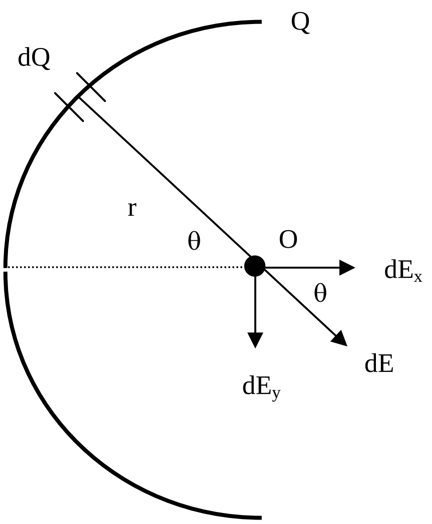
What happens as $a \to \propto$, i.e., as a gets very long?

$$\sqrt{\frac{x^2}{a^2}+1} \to 1 \therefore \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 x}\hat{i} \qquad \text{(for a long line charge)}$$

Notice that the dependence is $1/r$ instead of $1/r^2$. The direction is radial to the line charge distribution.

Example 7

Consider a line charge distribution, *Q*, distributed uniformly along length *L*. Bend the line into a semicircle as shown below. Find the electric field at point *O*.



By symmetry $E_y = 0$

$$dE = k\frac{dQ}{r^2}, dE_x = dE\cos\theta = \frac{kdQ}{r^2}\cos\theta$$

Try polar coordinates: *r*, θ

$dQ = \lambda dL = \lambda r d\theta$ (where $rd\theta$ is the arc length)

$$dE_x = \frac{k\lambda d\theta}{r}\cos\theta \therefore E_x = 2\frac{k\lambda}{r}\int_{\pi}^{\frac{\pi}{2}}\cos\theta d\theta$$

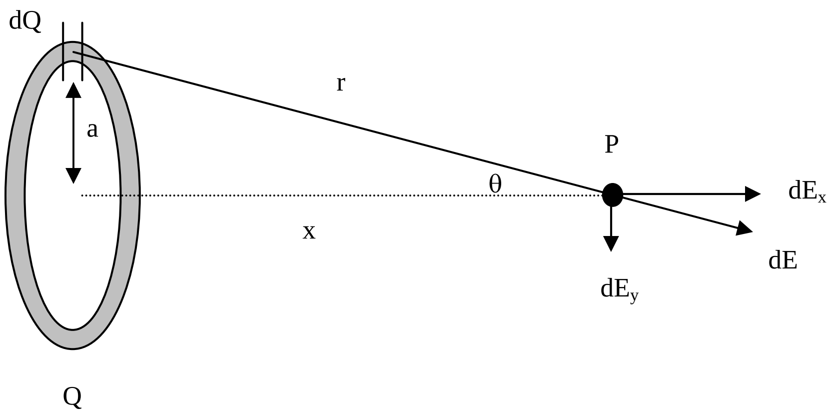
$$E_x = 2\frac{k\lambda}{r}\sin\theta_{\pi}^{\frac{\pi}{2}} = 2\frac{k\lambda}{r}$$

Note: $\lambda = \frac{Q}{L}, r = \frac{L}{\pi} \therefore \frac{\lambda}{r} = \frac{Q\pi}{L^2}$

$$\vec{E} = \frac{2kQ\pi}{L^2}\hat{i} = \frac{2k\lambda}{r}\hat{i}$$

Example 8

Consider a ring of charge, radius *a*, as shown below. Compute the electric field at point *P*.



Note: $E_y = 0$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{x^2 + a^2}$$

$$dE_x = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{x^2 + a^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{xdQ}{\left(x^2 + a^2\right)^{\frac{3}{2}}}$$

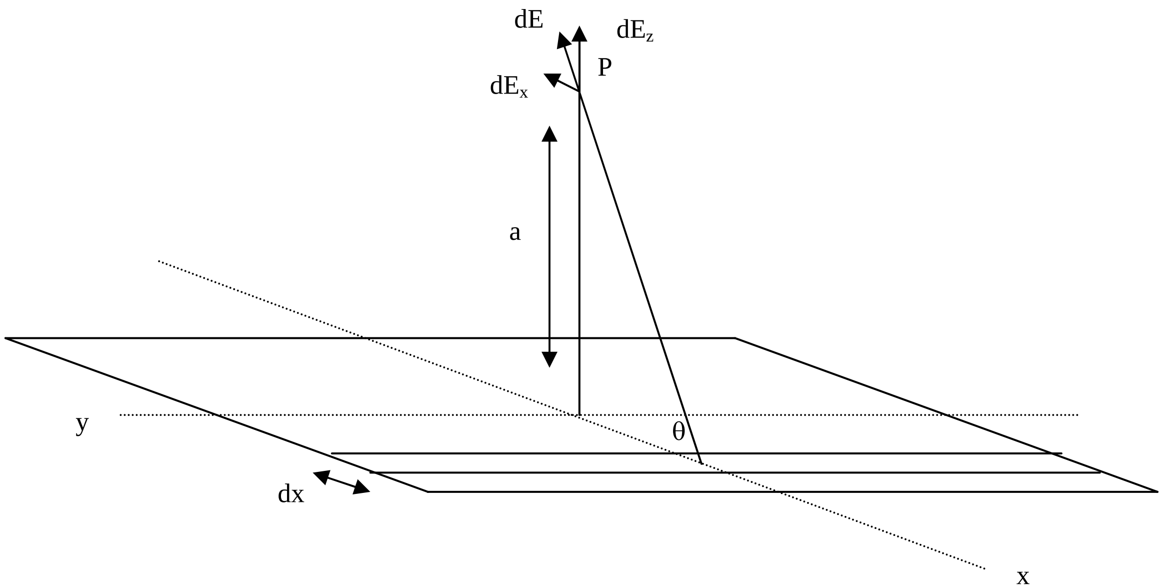
$$E = E_x = \frac{1}{4\pi\varepsilon_0}\int\frac{xdQ}{\left(x^2 + a^2\right)^{\frac{3}{2}}} = \frac{1}{4\pi\varepsilon_0}\frac{x}{\left(x^2 + a^2\right)^{\frac{3}{2}}}\int dQ$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{xQ}{\left(x^2 + a^2\right)^{\frac{3}{2}}}$$

Note: if $x = 0$, $E = 0$
if $a = 0$, $E = kQ/x^2$

Example 9

Consider an infinite sheet of uniform surface charge density σ. Compute the electric field at a point, *P*, located a distance, *a*, above the sheet.



The area of a strip of charge width *dx* and length *L* is *Ldx*. The charge, *dQ*, in this differential is: $dQ = \sigma L dx$.

Recall: $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ for a point *P* some distance, *r*, from a line charge distribution.

Note: $\lambda = \sigma dx \therefore E = \dfrac{\sigma x}{2\pi\varepsilon_0 r} \therefore dE = \dfrac{\sigma dx}{2\pi\varepsilon_0 r}$

Note: $E_x = 0 \therefore E = E_z$

$$dE = dE_z = dE\sin\theta \therefore E = \int_{-\infty}^{+\infty} dE_z = \frac{\sigma}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{\sin\theta dx}{r}$$

Note: $\sin\theta = \dfrac{a}{r}$, $r^2 = a^2 + x^2$

$$E = \frac{\sigma a}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{a^2 + x^2}$$

A definite integral (CRC #16)

$$E = \frac{\sigma a}{2\pi\varepsilon_0}\left[\frac{1}{a}\arctan\frac{x}{a}\right]_{-\infty}^{\infty}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

No dependence upon *a*, i.e., a uniform field.

Conductors and Insulators

Conductors and insulators are two broad classes of materials which my hold or transport charges.

The behavior of conducting and insulating materials runs across a continuum but we will, for the time being, consider them to be entirely separable and consider only the cases of absolute insulators and absolute conductors

- conductors allow the flow of charge
- charges move along the surfaces of all conductors
- insulators impede the flow of charge
- charges are evenly distributed in insulators

All charges reside on the surfaces of conducting materials. This, as we shall soon see, is very important in electric field computations. Insulators have charges uniformly distributed throughout.