Philip L. Cole
Idaho State University
Department of Physics

1 Vectors

1.1 Scalars and Units

Some quantities, such as temperature, mass, and time can be specified solely by a number with its associated unit. A quantity described purely by a number is called a **scalar**. For example, if someone were to ask you what time it is, in your reply, you would simply state a number with the appropriate units. If it were 10:04, you may say 4 minutes past ten o'clock. Furthermore scalars do not depend upon direction. If you were standing facing east in a room maintained at a constant temperature of 20°C, turning to the direction of due north is not going to alter the temperature. It does not matter whether pigs have wings, why the sun does not shine in the middle of the night, or what you ate for breakfast; no other information is relevant. All that is required is the pure number and its units. Remember specifying the unit is important. If you're in Mexico and someone tells you it is 20° outside, you do not have to put on an overcoat. Whereas 20°C makes for a comfortable temperature, 20° on the Fahrenheit scale is below the freezing point of water. In fact, there is a useful formula for converting degrees in Celsius to degrees in Fahrenheit:

$$T_{\mathrm{F}} = \frac{9}{5} T_{\mathrm{C}} + 32^\circ \tag{1}$$

Question 1: Solve this equation for T_{C} to obtain an expression for converting Fahrenheit to Celsius.

Question 2: (a) How many seconds are there in 13 hours, 14 minutes and six seconds? (b) How many minutes, and (c) how many days?

1.2 The Concept of a Vector

Other quantities such as *displacement* require additional information. Let's say, you are standing in the middle of a field with a blindfold on, which is bound tightly over your eyes and you cannot peek through the fabric. Someone has placed a briefcase loaded with lots of money a distance of 30 meters from you. You are told that if you get to the briefcase within 2 minutes, the briefcase and the contents therein are yours taxfree. If you only are told the distance of 30 meters, it is very unlikely that you will reach the briefcase in time to claim your reward. You would probably just wander around and around in dazed frustration as the time peters out. However, if you are given the instructions: make a quarter turn to your left (90°) and then proceed 30 meters, chances are you will stumble across the briefcase within the alloted period of time. What is different here? You were given a *direction* in space (a quarter turn to your left) and a measure of "how much, " i.e. the *magnitude* of 30 meters. Quantities involving both a magnitude and a direction are called **vectors**.

The operations for adding vectors are different than for scalars. For example, 1 hour + 1 minute is clearly equal to 61 minutes. However, how would you calculate the distance from the starting point O to the endpoint P?:

- A girl is initially standing at point O.
- She proceeds 3 meters due East
- She then turns 90° to her left, i.e. facing directly North.
- Thereafter she travels 4 meters due North.
- And finally she stops at point P.

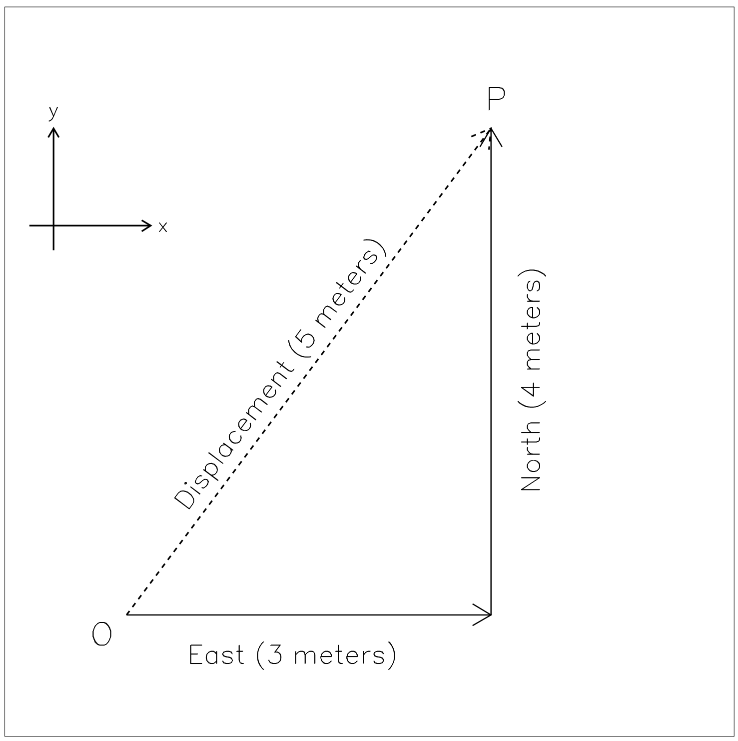

Figure 1: The girl's journey from point O to point P.

We have used the cartesian coordinate system (the x axis points towards the right and the y axis towards the top). In this case, the two subsequent displacements (3 meters east and 4 meters north) form the legs of a right triangle. The magnitude of the displacement (or the hypotenuse) from point O to point P is not equal to the distance traveled, i.e. 3 m + 4 m = 7 m. What we must do is decompose the vector into its components and then add these components in quadrature. This is just fancy language for expressing the concept of adding the squares of the length of these legs of the right triangle to obtain the square of the hypotenuse. This procedure is known as the *Pythagorean Theorem.*

$$r^2 = x^2 + y^2 \tag{2}$$

This mathematical discovery is named after the Greek philosopher Pythagorus, who lived in Sicily nearly 2500 years ago. As we see in Fig. 1, we have pointed East in the x direction and North in the y direction. If the magnitude of x is 3 meters and the magnitude of y is 4 meters, it then follows from the Pythorean Theorem that the magnitude of r is $\sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5$ m. We therefore have a 5-4-3 triangle, where the hypotenuse is of length 5 meters.

Question 3: A boy walks 50 meters directly south, he then proceeds a distance of x meters due west. (a) If the magnitude of the displacement r is 100 m from his starting point to his stopping point, how far has he traveled west? (b) How far would he have walked in total if he were to travel directly back to his starting point, i.e. what is the perimenter of this triangle?

1.3 Components of a Vector

So far we have only introduced the displacement in our discussion of vectors. Many any other quantities in the physical world are vectors as well, such as force, velocity, momentum, and the magnetic field. Symbolically we describe a vector quantity in **boldface type** or with an arrow above the quantity. For example, the vector quantity force is written either as **F** or $\vec{F}$, where the variable F stands for *force*. From now on we will write vector quantities with arrows over them, since it is much easier to write vectors in this manner. The magnitude of vector is the measure of "how much," and is written as $|\vec{F}|$. Mathematically, the magnitude of a vector is written as

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}, \tag{3}$$

where F_x and F_y are the x and y components of the vector quantity force, $\vec{F}$. Conceptually, this is identical to decomposing the displacement vector r into the components x and y, just like we did above.

CAUTION! It is absolutely imperative that YOU WRITE VECTOR QUANTITIES WITH THE ARROWS OVER THEM. Remember the briefcase full of unmarked bills in denominations of 20s and 50s described above? You'll never get to the goodies unless someone points you in the right direction. Just knowing the distance is not good enough. If you neglect to write the arrow over the variable, it means that you are unclear of the concept of the vector, which will lead to doing crazy things like adding the x and y components of a vector as if they were scalars instead of correctly using the Pythagorean Theorem. And that is not good.

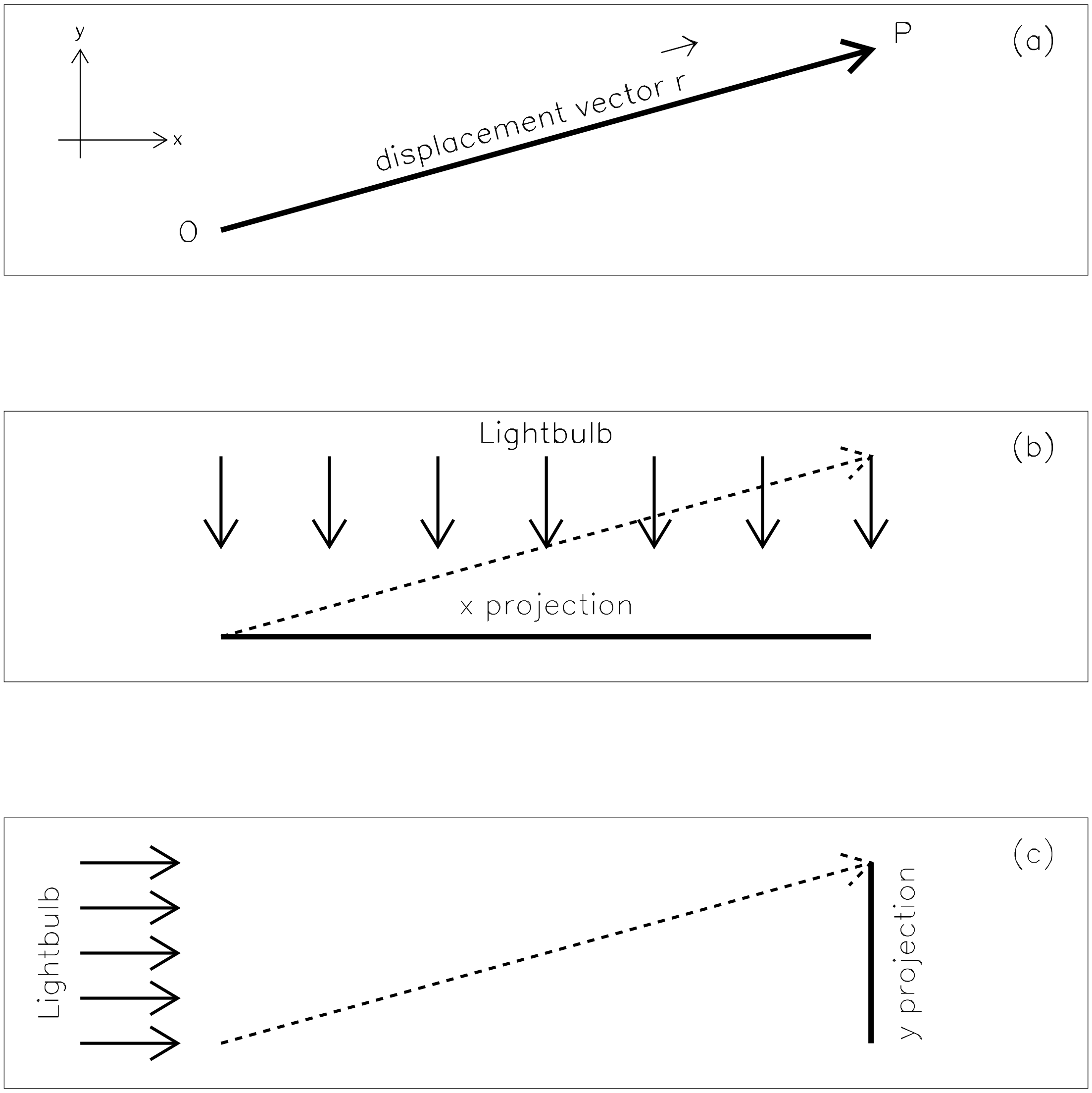

Figure 2: The x and y projections of the displacement vector $\vec{r}$.

1.4 Unit Vectors

A rather elegant way to write a vector is to employ the method of *unit vectors*. In planar geometry, i.e. two-dimensional space, we can decompose a vector into its x and y components. Let us describe this by the lightbulb, ruler and shadow analogy. In the cartesian coordinate system depicted in Fig. 2a we have a displacement vector $\vec{r}$ of length $|\vec{r}|$ extending from the origin O to point P. Let's assume this vector is a ruler. In Fig. 2b we have placed a lightbulb above the the ruler and parallel to the y-axis. The shadow it casts upon the ground is the *projection* of the ruler r onto the x axis. The length of this shadow is x. Likewise, if we shift the light source to the left of the ruler r and set it parallel to the x-axis, the shadow that is cast upon the y-axis is of length y, which is illustrated in Fig. 2c. If we add these two *projections* or *shadows* in quadrature, i.e. by means of the Pythagorean Theorem, we obtain $r^2 = x^2 + y^2$. This means the magnitude of vector $\vec{r}$ is $|\vec{r}| = \sqrt{x^2 + y^2}$.

Here's a fun experiment:

<u>EXPERIMENT I</u>

Equipment:

- string (red and green)
- rod
- rigid pivot
- ruler
- fishing line weights

Set the rod at an angle α with respect to the horizontal. Weight a length of red string with some lead. Hang the weighted red string from the tip of the rod. This weighted red string will point directly down towards the center of the earth. Mark the point where the red string intersects the table. Cut the red string to length from the point at which it drapes upon the table to the vertical elevation of the tip of the rod. Now take a piece of green string and unfurl it from the intersection point to the pivot point along the plane of the table and cut to length. You will now measure the two lengths of string. The length of red string (the hanging part) ℓ_{red}, and the green string is of length ℓ_{green}. You will find that the length of the rod is (or better be!)

$$\ell_{\text{rod}} = \sqrt{\ell^2_{\text{green}} + \ell^2_{\text{red}}} \tag{4}$$

Verify this. Measure the lengths of the red and green strings and the length of the rod. Does equation 4 hold? That is, can you retrieve the length of the rod by employing the Pythagorean Theorem?

Imagine we now take away the rod. We lay out the length of green string along the table. We will call this direction $\hat{\imath}$. We then lay the red string along the table in a direction perpendicular ($\perp$) to that of the green. We define the direction of the red string as $\hat{\jmath}$. These vectors with the pointy hats are called *unit vectors*. They are of unit length (i.e. of length 1) and point in the direction of the laid-out strings. In this case:

$$\vec{\ell}_{\rm rod} = \ell_{\rm green}\hat{\imath} + \ell_{\rm red}\hat{\jmath} \tag{5}$$

Does the rod form the hypotenuse of these two legs (i.e. the green and red strings). If it does (and it will if you did the experiment right!), then expression 5 is an alternate way of expressing a vector in terms of its components. In general we can write a vector as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}, \tag{6}$$

where the vector $\vec{r}$ can be decomposed into the components x and y, which are in the directions $\hat{\imath}$ and $\hat{\jmath}$, respectively. Futhermore, $\hat{\imath}$ and $\hat{\jmath}$ are perpendicular to one another, or $\hat{\imath} \perp \hat{\jmath}$. For the case of the vector quantity of force, we can write

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}, \tag{7}$$

or the *projection* of the force upon the x-axis added *vectorially* to the *projection* of the force upon the y-axis. Remember, the operative word is adding these x and y components *vectorially*. Again adding F_x to F_y as if they were scalars is nuts! Its like adding penguins to kumquats – it's meaningless. You must add the components of a vector in quadrature by using the Pythagorean Theorem.

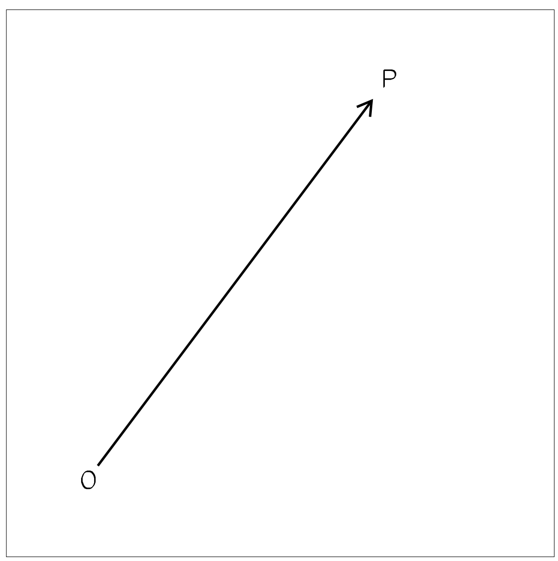

Figure 3: Graphically, a vector is depicted an arrow. The length of the arrow represents the *magnitude* of the vector, and the arrow *points* in the direction of the vector.

1.5 Adding Vectors

How does one go about adding two vectors? Let's say you wish to add the vector $\vec{A}$ to the vector $\vec{B}$. Unless $\vec{A}$ is perpendicular to $\vec{B}$, you cannot apply the Pythagorean Theorem, for this mathematical truth holds only for the legs of a right triangle, which are perpendicular to one another. In general, there are two ways to add vectors together: algebraically and graphically. We shall touch upon both methods in turn. For the algebraic method, we shall make use of the concept of the unit vector. As we saw earlier, that the vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$. Similarly we may write $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$. Although we may not add penguins to kumquats, nothing prevents us from adding penguins to penguins and kumquats to kumquats, since we are adding like quantities. Hence

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}. \qquad (8)$$

Here, we have added the x and y components of the vector $\vec{A}$ to those of vector $\vec{B}$.

Question 4: For the following, find $\vec{A} + \vec{B}$:

(a) $\vec{A} = 5\hat{\imath} - 6\hat{\jmath}$ and $\vec{B} = -5\hat{\imath} + 6\hat{\jmath}$
(b) $\vec{A} = 6\hat{\imath} - 5\hat{\jmath}$ and $\vec{B} = -5\hat{\imath} + 6\hat{\jmath}$
(c) $\vec{A} = 5\hat{\imath} - 5\hat{\jmath}$ and $\vec{B} = -5\hat{\imath} + 6\hat{\jmath}$
(d) $\vec{A} = 6\hat{\imath} - 6\hat{\jmath}$ and $\vec{B} = -5\hat{\imath} + 6\hat{\jmath}$.

Question 5: Now graph the vectors $\vec{A}$, $\vec{B}$, and $\vec{A} + \vec{B}$ on a cartesian coordinate system for parts (a) through (d) for Question 4.

If two vectors point in the same direction, they are said to be **parallel**. In Fig. 4a we draw the vectors $\vec{A} = 3\hat{\imath}$ and $\vec{B} = 4\hat{\imath}$. If we add these two vectors, we obtain a *new* vector of length 7 and pointing in the direction of the unit vector $\hat{\imath}$. In Fig. 4b we draw the vectors $\vec{A} = 3\hat{\imath}$ and $\vec{B} = -4\hat{\imath}$. In this case vector $\vec{B}$ is said to be **antiparallel** to vector $\vec{A}$, that is $\vec{B}$ points in the direction *directly opposite* that of $\vec{A}$. Adding $\vec{B}$ to $\vec{A}$ yields the quantity $-1\hat{\imath}$. In Fig. 4c we show the case of two vectors perpendicular to one another, $\vec{A} = 3\hat{\imath}$ and $\vec{B} = 4\hat{\jmath}$. The resultant vector $\vec{R}$ is equal to the vector sum of $\vec{A}$ and $\vec{B}$, where the length of vector $\vec{R}$ is $|\vec{R}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2} = 5$ units, and points in the direction of the angle α with respect to the horizontal. In general, two vectors need not identically point parallel, antiparallel or perpendicular to one another – they may point in any direction as is depicted in Fig. 4

We can add the two vectors graphically to obtain the resultant vector by the *head-to-tail* method. We have been using this *head-to-tail* method depicted in Figs. 4a–c without really calling it such. To represent a vector on a diagram we draw an arrow. We choose the length of the arrow to be proportional to the magnitude of the vector. We choose the direction of the arrow to be in the direction of the vector, with the arrowhead giving the direction of the vector. That is to say if we are to represent a vector $\vec{A}$ of length 3 m directed towards the north, on our graph we draw the vector. For example, a displacement of 3 m on a scale of 1 cm per 1 m, would be represented by an arrow of length 3 cm in the y direction. Here, we have set East and North to lie in the directions of x and y, respectively. We now take another vector $\vec{B}$ of length 4 m, which points in the direction of east. How do we geometrically add these two displacements? On a diagram drawn to scale we lay out the displacement of vector $\vec{A}$. We then draw $\vec{B}$ placing its tail at the head of $\vec{A}$. The line from the tail of vector $\vec{A}$ to to head of vector $\vec{B}$ constitutes the **vector sum** or **resultant** vector. We can express this relationship symbolically as $\vec{R}$

$$\boxed{\vec{R} = \vec{A} + \vec{B}}$$

And in the Fig. 5 we draw $\vec{R} = \vec{A} + \vec{B}$.

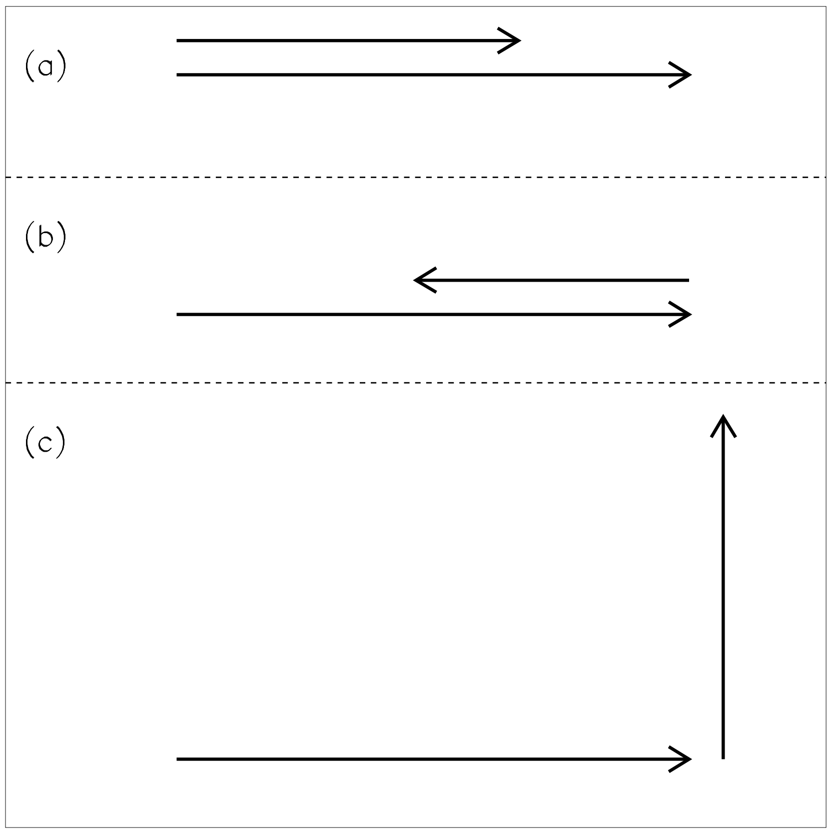

Figure 4: Two vectors which are (a) parallel, (b) antiparallel, (c) perpendicular to one another.

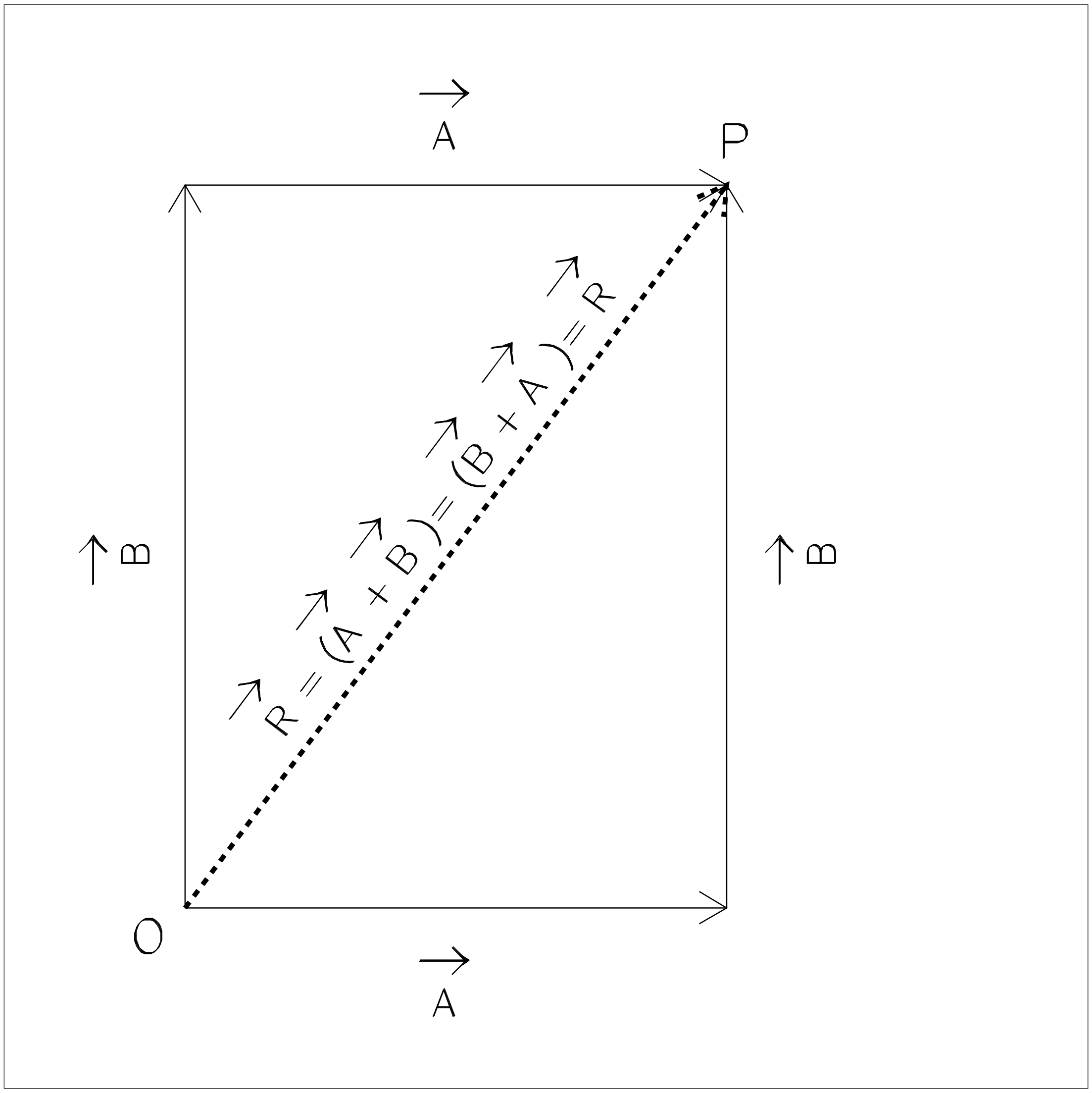

Figure 5: Vector $\vec{R}$ is the vector sum of vectors $\vec{A}$ and $\vec{B}$.

If we reverse the order of the displacements $\vec{A}$ and $\vec{B}$, with $\vec{B}$ first and $\vec{A}$ second, the result is the same

$$\vec{R} = \vec{B} + \vec{A}$$

This implies that the order in vector addition does not matter.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

In other words, vector addition obeys the *commutative law.* You can check out that the magnitude of the resultant vector is independent of the ordering. Let us assign values and directions to the vectors $\vec{A}$ and $\vec{B}$ by setting $\vec{A} = 4\hat{\imath}$ and $\vec{B} = 3\hat{\jmath}$. We find the magnitude of the resultant vector by employing the Pythagorean Theorem.

$$|\vec{R}| = \sqrt{B^2 + A^2} = \sqrt{3^2 + 4^2} = 5 = \sqrt{4^2 + 3^2} = \sqrt{A^2 + B^2}$$

Question 6: For the following, graph $\vec{A} + \vec{B}$:

(a) $\vec{A} = +5\hat{\imath}$ and $\vec{B} = -6\hat{\jmath}$
(b) $\vec{A} = +6\hat{\imath}$ and $\vec{B} = -5\hat{\imath}$
(c) $\vec{A} = -6\hat{\jmath}$ and $\vec{B} = +5\hat{\imath}$
(d) $\vec{A} = +6\hat{\imath}$ and $\vec{B} = -6\hat{\imath}$.

Multiplying a vector with a scalar quantity yields a physical quantity having units that are a product of the units associated with the scalar and the vector. From Newton's second law, $\vec{F} = m\vec{a}$; the net force acting on a body is the product of the mass m (scalar) of that body and its acceleration $\vec{a}$ (vector). The unit of the magnitude of the force is the unit of mass multiplied by the unit of the magnitude of the acceleration the body experiences. The direction of the force $\vec{F}$ is always in the same the direction of the acceleration $\vec{a}$, because mass is a positive quantity.

1.6 Trigonometry

The geometrical method for adding vectors becomes very cumbersome if one needs to add more than two vectors, and this method loses its utility once one extends to three dimensions. A very useful approach is the *analytical method*, which entails resolving the vectors into components with respect to a particular coordinate system. First we shall study how to resolve a vector into its x and y components. Let us draw a vector $\vec{R}$ having an angle of 36.9° with respect to the horizontal of length 5 units. This vector is drawn below.

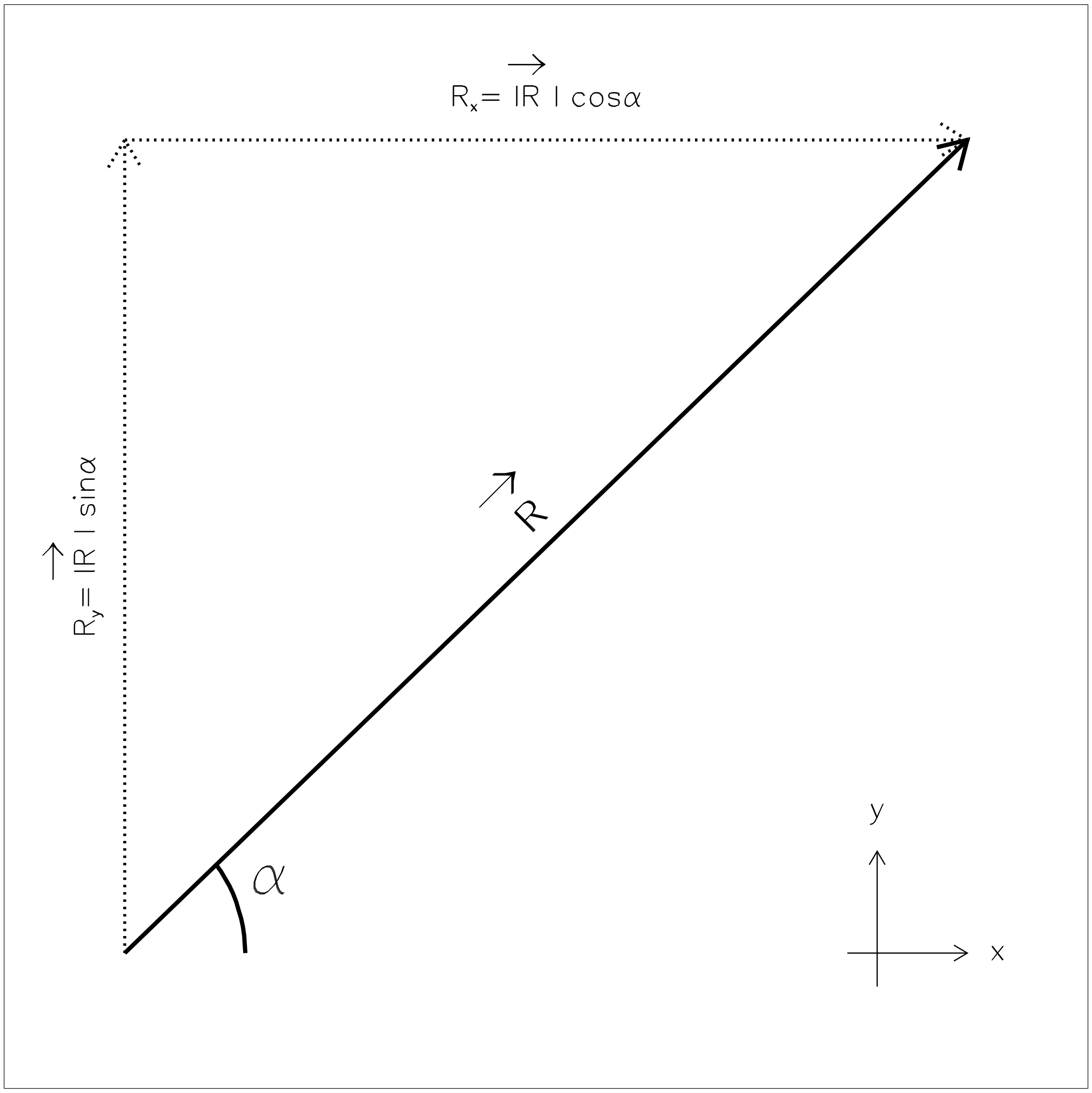

Figure 6: An example of resolving a vector into its components.

One readily sees in Fig. 6 that the height (or distance along the y-axis) is 3 units, and the length (or distance along the x-axis) is 4 units. We have our recovered our old friend the 5-4-3 triangle. If we specify the length of vector $\vec{R}$ and the angle the vector makes with respect to the horizontal, we can uniquely determine its x and y components by using *trigonometry*. We shall use the trigonometry functions cosine and sine to determine the projections onto the x and y-axis for the vector $\vec{R}$ that makes an angle α with respect to the x-axis. We define:

$$R_x = |\vec{R}| \cos(\alpha) \leftarrow x \text{ component of vector } \vec{R}$$

Here, $\cos(\alpha) = R_x/R$.

$$R_y = |\vec{R}| \sin(\alpha) \leftarrow y \text{ component of vector } \vec{R}$$

Here, $\sin(\alpha) = R_y/R$

We observe that

$$-1 \leq \cos(\alpha) \leq +1 \text{ and } -1 \leq \sin(\alpha) \leq +1$$

This is to say that the absolute value of the sine or cosine of an angle cannot exceed unity. We also observe that adding the square of the cosine of an angle to the square of the sine of that same angle yields unity.

$$\cos^2(\alpha) + \sin^2(\alpha) = (R_x/R)^2 + (R_y/R)^2 = \frac{R_x^2 + R_y^2}{R^2} = \frac{R^2}{R^2} = 1$$

We define the tangent of an angle to be:

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{(R_y/R)}{(R_x/R)} = (\frac{R_y}{R})(\frac{R}{R_x}) = \frac{R_y}{R_x}$$

Question 7: Using your calculator find $\sin(\alpha)$ and $\cos(\alpha)$ to 3 significant figures for the following angles. (make sure your calculator is in degree mode): (a) $\alpha = 0°$; (b) $\alpha = 45°$; (c) $\alpha = 90°$; (d) $\alpha = 135°$; (e) $\alpha = 180°$; (f) $\alpha = 225°$; (g) $\alpha = 270°$; (h) $\alpha = 315°$; (i) $\alpha = 360°$; (j) $\alpha = 36.87°$; (k) $\alpha = 30°$; (l) $\alpha = 60°$.

Question 8: Show that $\cos^2(\alpha) + \sin^2(\alpha) = 1$ for parts (a) through (l) in Question 7.

Question 9: Show that $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ for parts (a) through (l) in Question 7.

Question 10: For the following use the relationships $R_x = |\vec{R}| \cos(\alpha)$ and $R_y = |\vec{R}| \sin(\alpha)$. Given that $|\vec{R}| = 5$, calculate R_x and R_y for the angles given in parts (a) through (l) in Question 7, and show Show that $|\vec{R}| = \sqrt{R_x^2 + R_y^2}$.

Now, let us take advantage of the power of the analytical method to calculate the vector sum of two (or more) vectors. Suppose we wish to calculate the resultant vector $\vec{R}$ from the vector sum of $\vec{A}$ and $\vec{B}$. How would we calulate the magnitude and direction of $\vec{R}$ if vector $\vec{A}$ were to make an angle of 30° with respect to the x-axis and had a magnitude of 4 units and vector $\vec{B}$ made an angle of 60° with respect to the x-axis and had a magnitude of 5 units. In Fig.7, we graph two vectors $\vec{A}$ and $\vec{B}$ and their vector sum $\vec{R}$, along with the x- and y-components of all three vectors.

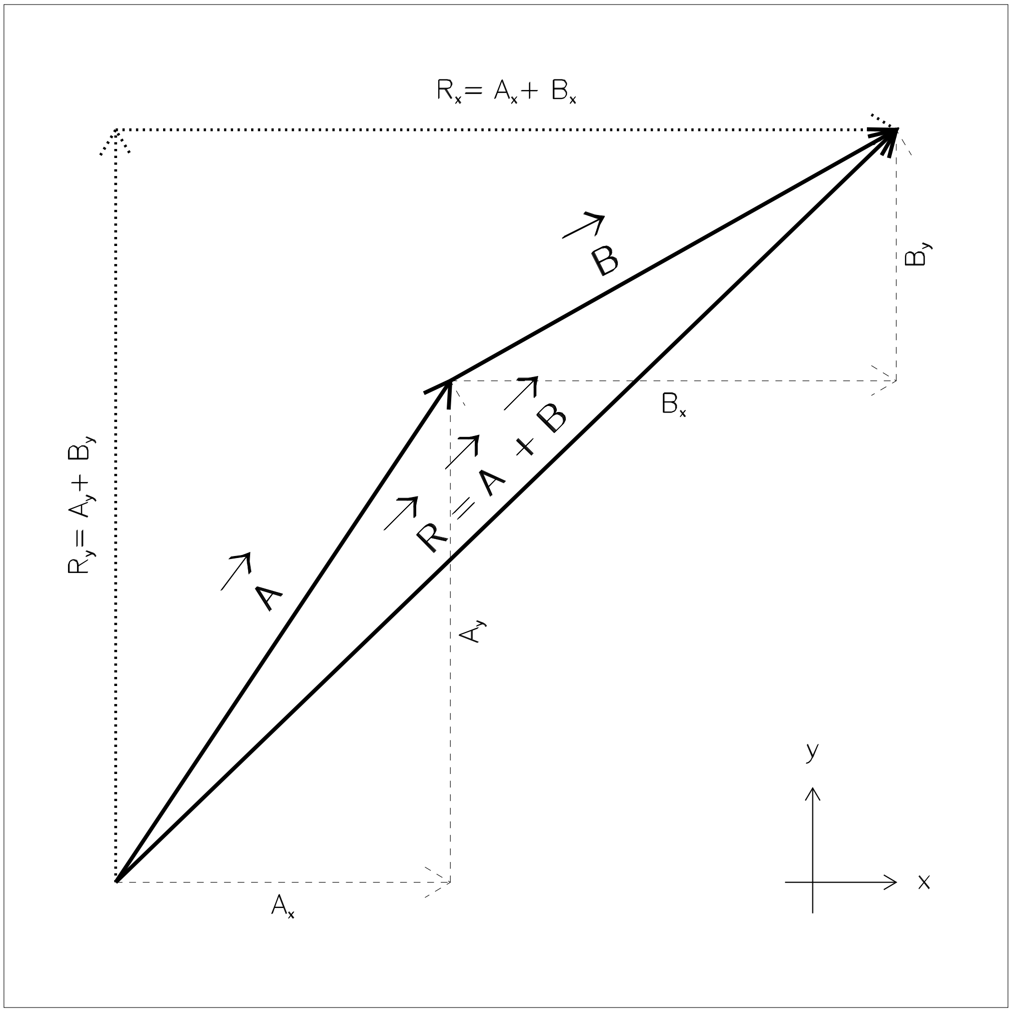

Figure 7: A graphical example of decomposing the sum of two vectors, $\vec{A}+\vec{B}$ into their respective x and y components. Here, $R_x = A_x + B_x$ and $R_y = A_y + B_y$. The magnitude of the resultant vector, $|\vec{R}|$, is then $\sqrt{R_x^2 + R_y^2}$.

The reader can readily observe from this figure that the x-component R_x of the vector sum is simply the sum of the x-components of the vectors $\vec{A}$ and $\vec{B}$. The same is true for the y-components. In symbols the components of $\vec{R} = \vec{A} + \vec{B}$:

$$R_x = A_x + B_x \text{ and } R_y = A_y + B_y$$

Because $A_x = |\vec{A}|\cos(\theta_1)$, $A_y = |\vec{A}|\sin(\theta_1)$, $B_x = |\vec{B}|\cos(\theta_2)$, and $B_y = |\vec{B}|\sin(\theta_2)$, we can readily calculate the vector sum.

Question 11: Let the vector $\vec{A}$ have a magnitude of 4 units and make an angle of 30° with respect to the x-axis. And let the vector $\vec{B}$ have a magnitude of 4 units and make an angle of 30° with respect to the x-axis. Calculate (a) the x and y components of the resultant vector, (b) the magnitude of the resultant vector, and (c) the angle the resultant vector makes with respect to the x-axis.

Question 12: Graph the results of Question 11.

We can easily extend this analytical procedure for finding the sum of two vectors to any number of vectors. Let the resultant vector $\vec{R}$ be the vector sum of $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, ... The components of $\vec{R}$ are then:

$$R_x = A_x + B_x + C_x + D_x + E_x + \cdots$$

$$R_y = A_y + B_y + C_y + D_y + E_y + \cdots$$

Question 13: Let $\vec{R} = \vec{A} + \vec{B} + \vec{C}$. $R_x = 1$, $A_x = 2$, $R_x = 3$, $R_y = 5$, $A_y = 3$, and $R_y = 2$. Calculate: (a) C_x and C_y. (b) The angles the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{R}$ make with respect to the x-axis. (c) The magnitudes of these vectors.

The mathematics of resolving vectors into component form was established because many physical quantities are vectors. In science and engineering the language is mathematics. To become a successful and well-paid engineer, you must become fluent in this language. You should take this opportunity to review the chapter on vectors. Also if you have any questions, please ask your instructor those concepts that you find confusing – professors like it when students ask questions!