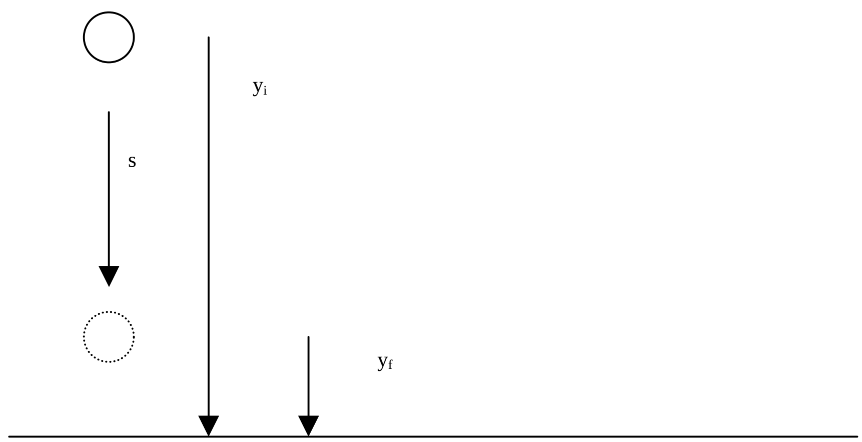
# Potential Energy/Conservation of Energy/ Non-conservative Forces

**Potential Energy -** In the same manner as Kinetic Energy may be thought of as the energy of motion, *Potential Energy* may be thought of as the energy of *position*. Potential Energy may take many forms but is commonly of one of the following types:

- Gravitational
- Electrical
- Elastic




Absolute potential energy, in general, is not a very useful concept. One generally speaks of the *change* in potential energy within a system rather than absolute potential energy. Since all potential energy is relative to some point of reference, it is often convenient to define a reference level at which potential energy is zero.

**Gravitational P.E.**

Consider the work done by the gravitational force when an object falls to earth.

$$W_{grav} = F_g \cdot s = mg \cdot s = -(mg)\hat{j} \cdot (y_f - y_i)\hat{j} = mgy_i - mgy_f$$

Define: $U_g \equiv mgy$ as gravitational potential energy.

$$W_{grav} = U_{g_i} - U_{gf}$$

Notice that the work done by gravitational potential energy depends only upon the initial and final coordinates, not the path taken. This is because gravity is a conservative force.

**Elastic P.E.**

$$U_{spring} = \frac{1}{2}kx^2$$

**Conservation of Mechanical Energy**

When only *conservative* forces are present, total mechanical energy is conserved, i.e., $E = K + U$, or:

$$K_i + U_i = K_f + U_f \quad (1)$$

This means that in conservative systems mechanical energy is neither gained nor lost but merely changes form, e.g., the conversion of potential energy to kinetic energy as a raindrop falls toward the ground.

Since the only two types of mechanical energy are potential and kinetic:

$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f \quad \text{(gravity)}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 \quad \text{(spring)}$$

Conservative forces are defined as forces for which equation (1) holds true.

**Conservation of Mechanical Energy with non-conservative forces**

When non-conservative forces are present, total mechanical enery is not conserved, i.e., the total energy of the system changes ($E_i \neq E_f$).

**Applied Forces**

Suppose you heave an object from the ground into the air. As you lift the object, and until the moment it leaves your hands at some height above the ground, you do work on it. At the same time gravity also does work on the ball. According to the Work-Energy Principle:

$$W_{app} + W_g = \Delta KE$$

Because gravity is a conservative force, $W_g = -\Delta U$ (what is the significance of the minus sign here?). Hence:

$$W_{app} - \Delta U = \Delta KE \therefore W_{app} = \Delta KE + \Delta U \Rightarrow W_{app} = \Delta E$$

In this case the change in energy is to a system that includes both the earth and the object. The total energy was increased by increasing the distance between the ground and the object (increasing it's potential energy) and increasing its speed as it is lifted (increasing it's kinetic energy).

**Frictional Forces**

Frictional forces result in energy being *dissipated* from systems where friction is involved (this is a fancy way of saying that although we can compute the magnitude of the energy loss we are a bit cavalier, at least at this point, about exactly where it goes). In general, frictional forces reduce the kinetic energy of a system, i.e.:

$$W_f = \vec{f}_k \cdot \vec{d} = \Delta KE$$

or

$$W_f = \vec{f}_k \cdot \vec{d} = \Delta E$$

**Power**

Power is the rate of energy transfer. The S.I. unit of power is the *watt*, which is equivalent to a Joule per second.

$$\overline{P} = \frac{\Delta E}{\Delta t}$$

$$P = \frac{dE}{dt}$$

**Mass-energy equivalence**

Energy and matter are the same thing. When we study modern physics late next semester we will explore this bridge in more detail. Einstein's famous equation relates mass and energy:

$$E = mc^2$$

**Quantization of Energy**

As we will see later, all values of energy within a system are *not* possible, i.e., only certain discrete levels of energy are permitted within any system. For very large systems (things we deal with in our daily lives) the permitted energy levels are so close together that they appear to be a continuum. For atomic-sized systems, however, quantization of energy levels is readily apparent and extremely significant.

**Example 1.** Compute the work involved in walking to the top of a 1200 meter peak (this is about the elevation involved in hiking to the top of Scout Mountain from Justice Park).

For an average person, $m \approx 80kg$. This is an example of an applied force (walking) doing work.

$$W_{app} = \Delta E = \left(784kg \cdot m \cdot s^{-2}\right)\left(1200m - 0m\right) = 940800J$$

**Example 2.** Use conservation of total mechanical energy to compute the final velocity of an object dropped from some height, *h* just before it strikes the ground.

Since only conservative forces are present: $K_i + U_i = K_f + U_f$, or

$$\frac{1}{2}mv_i^{\,2} + mgh_i = \frac{1}{2}mv_f^{\,2} + mgh_f$$

If we define the ground as being at height $h = 0$, (zero potential energy reference level), then $h_i = h$ and conservation of mechanical energy proceeds as follows:

$$\begin{aligned} 0 + mgh &= \frac{1}{2}mv_f^{\,2} + 0 \\ mgh &= \frac{1}{2}mv_f^{\,2} \\ v_f &= \sqrt{2gh} \end{aligned}$$

**Example 3.** To what height will a ball rise if tossed upward with an initial velocity of 10 m/s?

Again, since only conservative forces are present: $K_i + U_i = K_f + U_f$ , or $\frac{1}{2}mv_i^{\ 2} + mgh_i = \frac{1}{2}mv_f^{\ 2} + mgh_f$ Once again we define the ground as being at height *h* = 0:

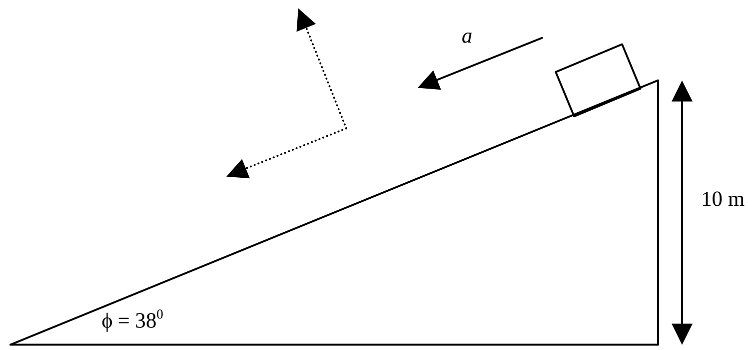
$$\begin{aligned} 0 + gh_f &= \frac{1}{2}v_i^{\ 2} + 0 \\ h_f &= \frac{v_i^{\ 2}}{2g} \\ h_f &= 5.1m \end{aligned}$$

We could also use Work-Energy:

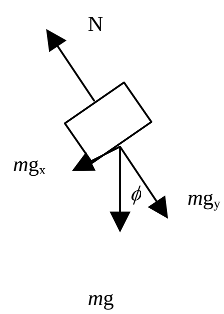
$$\begin{aligned} W = \Delta KE = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_i^{\ 2} &= m\vec{g} \cdot \vec{h} \\ 0 - \frac{1}{2}v_i^{\ 2} &= -gh \\ h_f &= 5.1m \end{aligned}$$

**Example 4.** A 10 kg block slides down a smooth surface. It is released from rest at the top of an incline ($38^0$ to the horizontal) at a height 10 meters. What is its speed at the bottom of the incline?

We've seen this problem before:




Note: $\sin 38^0 = \dfrac{10m}{hyp} \Rightarrow$ the length of the incline is 16.2 meters. In the coordinate system we've chosen, the FBD is:




Based on this FBD: $\sum F_x = mg_x = ma = mg\sin\phi \therefore a = g\sin\phi =$ 6.03m/s$^2$

By determining the acceleration we've solved the *dynamics* part of the problem and with this information we can proceed to solve the *kinematics* part of the problem.

$$v^2 = v_0{}^2 + 2a(x - x_0) \rightarrow v = \sqrt{2a(x - x_0)} = \sqrt{(2)(6.03m \cdot s^{-2})(16.2m)} = 14m \cdot s^{-1}$$

This method of solving the problem yields a solution but it depends upon resolving vectors, summing forces, determining acceleration, and finally kinematics!

Now consider using conservation of mechanical energy to address the same problem: Since only conservative forces are present:

$$K_i + U_i = K_f + U_f \text{ , or } \frac{1}{2}mv_i^{\,2} + mgh_i = \frac{1}{2}mv_f^{\,2} + mgh_f$$

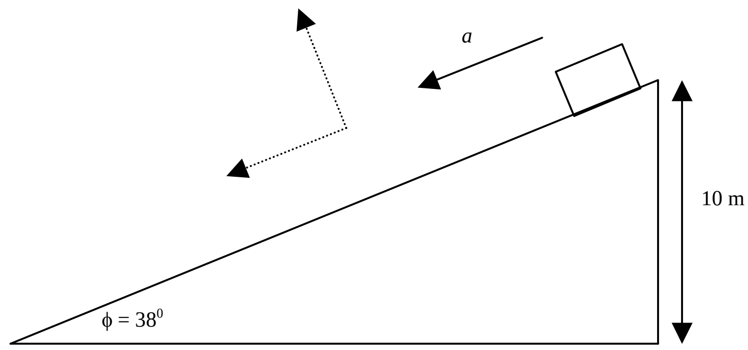
If we define the ground as being at height *h* = 0, (zero potential energy reference level), then $h_i = h$

$$0 + mgh = \frac{1}{2}mv_f^{\,2} + 0$$
$$mgh = \frac{1}{2}mv_f^{\,2}$$
$$v_f = \sqrt{2gh}$$
$$v_f = 14m \cdot s^{-1}$$

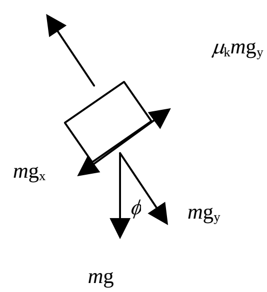
This is a much more compact and elegant method of solution.

**Example 5**: A 10 kg block slides down a rough surface ($\mu_k$=0.2). It is released from rest at the top of an incline ($38^0$ to the horizontal) at a height 10 meters. What is its speed at the bottom of the incline?

We've seen this problem before as well.




Note: $\sin 38^0 = \dfrac{10m}{hyp} \Rightarrow$ the length of the incline is 16.2 meters. In the coordinate system we've chosen, the FBD is:




Based on this FBD: $\sum F_x = mg\sin\phi - \mu_k mg\cos\phi = ma \therefore a = g\sin\phi - \mu_k g\cos\phi$

The first term is the same expression as we got when working this example without accounting for the presence of friction. The second term is a component that reduces the overall acceleration due to the presence of kinetic friction.

In this case:

$$a = \left(9.8m\cdot s^{-2}\right)(0.616) - (0.2)\left(9.8m\cdot s^{-2}\right)(0.788) = \left(6.03m\cdot s^{-2}\right) - \left(1.54m\cdot s^{-2}\right) = 4.5m\cdot s^{-2}$$

By determining the acceleration we've solved the *dynamics* part of the problem and with this information we can proceed to solve the *kinematics* part of the problem.

$$v^2 = v_0^{\ 2} + 2a(x - x_0) \rightarrow v = \sqrt{2a(x-x_0)} = \sqrt{(2)\left(4.5m\cdot s^{-2}\right)(16.2m)} = 12.1m\cdot s^{-1}$$

Now consider using conservation of energy. Since a non-conservative force is present (friction) $W_f = \vec{f}_k \cdot \vec{d} = \Delta KE$ or $W_f = \vec{f}_k \cdot \vec{d} = \Delta E$. Hence:

$$W_f = \left(\frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2}\right) + \left(mgh_f - mgh_i\right) = \vec{f}_k \cdot \vec{d}$$

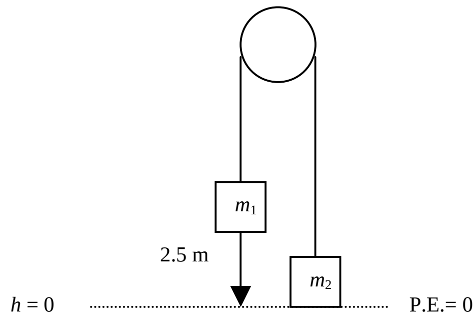
$$W_f = \frac{1}{2}mv_f^{\,2} - mgh_i = -\mu_k mg\cos\theta d$$

$$v_f = \sqrt{2gh - \left(2\mu_k g\cos\theta\right)d}$$

$$v_f = 12.1 m \cdot s^{-1}$$

Again, this is a much more compact method of solution.

**Example 6.** Consider a light, frictionless pulley with two masses $m_1$ & $m_2$ attached by a light cord as shown below. If $m_1$ = 5 kg and $m_2$ = 3.5 kg, determine the speed of the system when the 5 kg mass has fallen, starting at rest, a distance of 2.5 meters.




If we note, however, that the only force acting on this system (gravity) is conservative:

$$K_i + U_i = K_f + U_f$$

Hence:

$$\frac{1}{2}m_1 v_i^{\ 2} + \frac{1}{2}m_2 v_i^{\ 2} + m_1 g h_i + m_2 g h_i = \frac{1}{2}m_1 v_f^{\ 2} + \frac{1}{2}m_2 v_f^{\ 2} + m_1 g h_f + m_2 g h_f$$

$$\frac{1}{2}(m_1 + m_2)v_i^{\ 2} + m_1 g h_i + m_2 g h_i = \frac{1}{2}(m_1 + m_2)v_f^{\ 2} + m_1 g h_f + m_2 g h_f$$

$$m_1 g h_i = \frac{1}{2}(m_1 + m_2)v_f^{\ 2} + m_2 g h_f$$ ($v_i$ = 0, zero P.E. @ $h$ = 0)

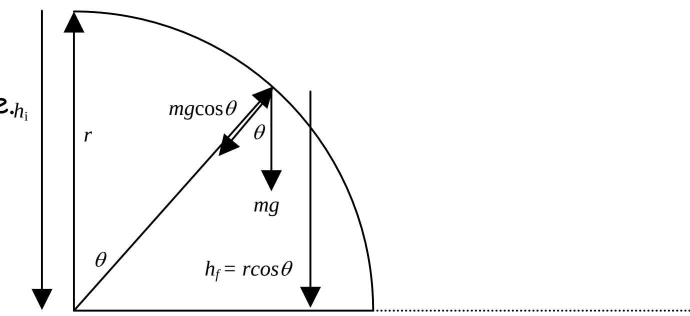
$$m_1 g h - m_2 g h = \frac{1}{2}(m_1 + m_2)v_f^{\ 2}$$ ($h_i$ = $h_f$ = $h$)

$$v_f = \sqrt{\frac{2(m_1 - m_2)gh}{m_1 + m_2}}$$

$$v_f = 2.94 m \cdot s^{-1}$$

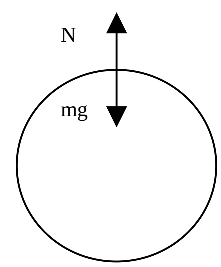
**Example 7.** An person starts from rest at the top of a large spherical surface and slides into a pool of water below. At what angle, $\theta$, does the person leave the surface of the slide?

Note: The person leaves the surface of the slide at the instant the normal force is equal to the centripetal force. After leaving the slide the normal force goes to zero. In this case it is *not* necessary to define P.E. = zero at $h = 0$. This problem is similar to Example 3 in that it asks for final position rather than final velocity, but does so, in this case, in terms of an angle




At the top of the slide, before the person begins their journey, the force of gravity acting downward is balanced by the normal force acting upward.

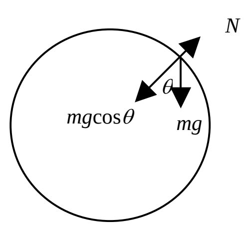
$$\sum F = N - mg \therefore N = mg$$




As the person moves down the slide, a component of gravity, *mg*cos$\theta$, supplies the centripetal force. The normal force is:

$$mg\cos\theta - N = \frac{mv^2}{r}$$

$$N = mg\cos\theta - \frac{mv^2}{r}$$



Finally, at the instant the person leaves the surface of the slide, $N = 0$:

$$\frac{mv^2}{r} = mg\cos\theta$$

$$v^2 = gr\cos\theta$$

Conservation of Mechanical Energy:

$$mgh_i = \frac{1}{2}mv^2 + mgh_f$$

$$mgr = \frac{1}{2}mv^2 + mgr\cos\theta \qquad \text{(in terms of } r, \theta\text{)}$$

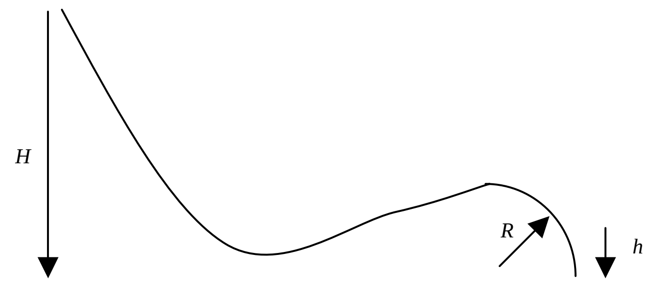
$$mgr = \frac{1}{2}mgr\cos\theta + mgr\cos\theta \qquad (v^2 = gr\cos\theta)$$

$$1 = \frac{1}{2}\cos\theta + \cos\theta = \frac{3}{2}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48^0$$

An interesting question to ask next is where does the person land in the water? The person's path changes from uniform circular to parabolic at the moment they leave the surface of the slide. In order to determine the range we need to know $\boldsymbol{h_f}$ and $\boldsymbol{v_f}$. How would you determine these quantities from the answer above?

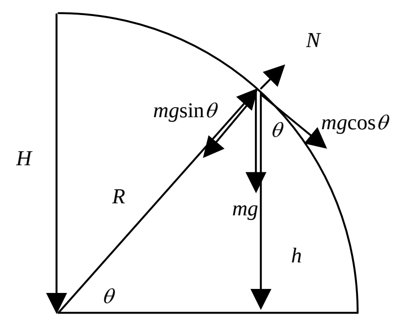
**Example 8.** A skier slides down a smooth slope starting from rest as shown below. In terms of the initial height, *H*, at what height, *h*, will they lose contact with the bump, radius *R*?




$E_i = E_f$ Conserve Mechanical Energy

$$\frac{1}{2}mv_i^{\ 2} + mgH = \frac{1}{2}mv_f^{\ 2} + mgh$$

$v_f^{\ 2} = 2g(H-h)$ ($v_i$ = 0)




We need to cast *h* in terms of *H*

$$mg\sin\theta - N = \frac{mv^2}{R}$$

$$N = mg\sin\theta - \frac{mv^2}{R}$$

At the instant the person leaves the surface of the bump, *N* = 0:

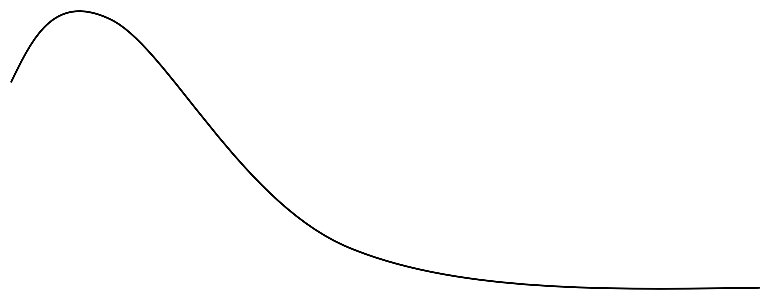
$$\frac{mv^2}{r} = mg\sin\theta$$

$mg\frac{h}{R} = mg\frac{2(H-h)}{R}$ (Note: $\sin\theta = \frac{h}{R}$)

$$h = 2(H-h)$$

$$h = \frac{2}{3}H$$

**Example 9.** An 80 kg skier starts from rest at the top of a 160 meter tall hill and attains a velocity of 36 m/s by the time they reach the bottom of the 200 meter run. What is the coefficient of kinetic friction between their skis and the snow?



A non-conservative force, friction, is present here, $E_i \neq E_f$, and total mechanical energy is not conserved.

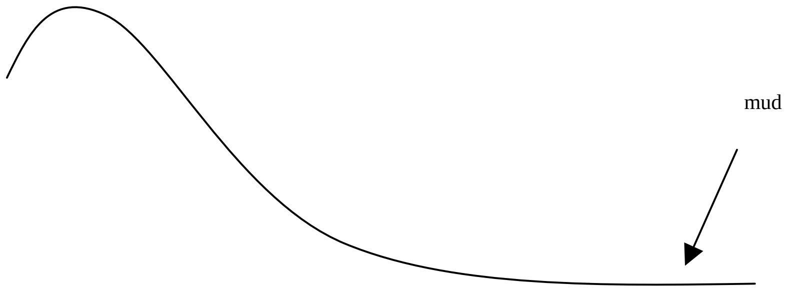
$E_i$ = $mgh$ = (80 kg)(9.8 m/s$^2$)(160 m) = 125440 J
$E_f$ = $\frac{1}{2}mv^2$ = (0.5)(80 kg)(36 m/s)$^2$ = 51840 J

$$W_f = \Delta E = -73600J = \vec{f}_k \cdot \vec{s} = -\mu_k mgs$$

$$\frac{73600J}{(200m)(80kg)(9.8m \cdot s^{-2})} = \mu_k = 0.47!$$

What'd they use for wax? The numbers in this problem are fairly realistic (except for the result). What did we fail to take into account? (Hint: Aside from the possible presence of other resistive forces, what about the force of kinetic friction, is it indeed constant?)

**Example 10.** A skier starts from rest at the top of a 160m tall hill (common in the midwest). The portion of the hill covered with snow is essentially frictionless but flat section at the base of the hill covered with mud (also common in the midwest) is not. If the skier weighs 80kg, and the coefficient of kinetic friction between rental skis and mud is 0.8, how much distance is required for the skier to come to rest?




A non-conservative force, friction, is present here, $E_i \neq E_f$, and total mechanical energy is not conserved.

$E_i$ = $mgh$ = (80 kg)(9.8 m/s$^2$)(160 m) = 125440 J
$E_f$ = 0

$$W_f = \Delta E = -125440J = \vec{f}_k \cdot \vec{s} = -\mu_k mgs$$

$$\frac{125440J}{(0.8)(80kg)(9.8m \cdot s^{-2})} = s = 200m$$

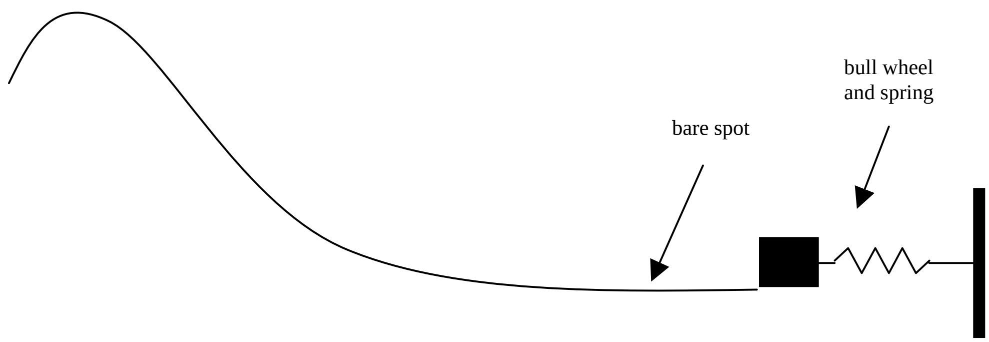
We could arrive at the same result by looking at the skier's velocity at the base of the hill and computing the change in *kinetic energy* to get the work done by the non-conservative force.

$$v_{bottom} = \sqrt{2gh} = 56m \cdot s^{-1}$$

$$v_{final} = 0$$

$$\Delta KE = -125440J \therefore s = 200m$$

**Example 11.** This time our hapless 80 kg skier starts from rest at the top of a 160 meter tall hill. The downhill portion of the run is snow covered and essentially frictionless but a flat section at the base of the hill has a bare spot 2 meters in length. The skier skis straight down the hill, over the bare spot at the bottom, hits the counterweight on the bullwheel compressing the damping spring 50 centimeters in the process. If the spring constant is $1 \times 10^6$ N/m, what is the coefficient of kinetic friction between rental skis and bluegrass?




A non-conservative force, friction, is present here, $E_i \neq E_f$, and total mechanical energy is not conserved.

$E_i = mgh$ = (80 kg)(9.8 m/s$^2$)(160 m) = 125440 J
$E_f = \frac{1}{2}kx^2$ = (0.5)(1 × 10$^6$ N/m)(0.5 m$^2$) = 125000 J

$$W_f = \Delta E = -440J = \vec{f}_k \cdot \vec{s} = -\mu_k mgs$$

$$\frac{440J}{(2meters)(80kg)(9.8m \cdot s^{-2})} = \mu_k = 0.28$$