Linear Momentum/Conservation of Momentum/Collisions/Energy

Linear Momentum

- The definition of linear momentum is: $\vec{p} = m\vec{v}$
- Since mass is an intrinsically positive quantity, ***p*** and ***v*** always point in the same direction (in the direction of motion).
- The S.I. unit of momentum is a kg · m · s^{-1}.

Newton's second law may be expressed (and was originally expressed) in terms of the time rate change of momentum or:

$$F = \frac{\Delta\vec{p}}{\Delta t}$$

The equivalence of the two expressions may be easily shown:

$$\frac{\Delta\vec{p}}{\Delta t} = \frac{\Delta(m\vec{v})}{\Delta t} = m\frac{\Delta\vec{v}}{\Delta t} = m\vec{a} \therefore F = m\vec{a}$$

For a system consisting of more than one particle, the total momentum, ***P***, of the system is:

$$\vec{P} = M_{total}\vec{v}_{centerofmass}$$

Conservation of Linear Momentum

Consider our definition of linear momentum, $F = \frac{\Delta\vec{p}}{\Delta t}$. Suppose that the sum of all external forces acting on a particle or system of particles is zero. In this case, the momentum of the system cannot be changing in time since $\frac{\Delta\vec{p}}{\Delta t} = 0$. When the sum of all external forces acting on an isolated system is zero, the total momentum of a system cannot change and momentum is said to be *conserved*, i.e.,

$$\vec{P}_i = \vec{P}_f$$

or

$$m\vec{v}_i = m\vec{v}_f$$

Caveat! Linear momentum is a vector quantity. What do you suppose is the consequence of external forces acting on a system that have no components acting in the direction of the velocity of the particle(s) in the system?

Conservation of linear momentum (or not)

- A ball bouncing off a wall?
- Firing a gun?
- A collision between two billiard balls?

Example 1. A 4.5 kg shotgun fires a 0.1 kg slug with a muzzle velocity of +150 m/s. What is the recoil velocity of the shotgun?

There is an external force that acts on this system, gravity. But in the scale of this problem gravity acts only perpendicularly to the velocities of various parts of the system so it has no discernable effect on the momentum along a horizontal axis.

$$\vec{P}_i = \vec{P}_f$$

$$m\vec{v}_i = m\vec{v}_f$$

$$m_{shotgun} v_{shotgun_i} + m_{slug} v_{slug_i} = m_{shotgun} v_{shotgun_f} + m_{slug} v_{slug_f}$$

$$0 + 0 = m_{shotgun} v_{shotgun_f} + m_{slug} v_{slug_f}$$

$$m_{slug} v_{slug_f} = -m_{shotgun} v_{shotgun_f}$$

$$\frac{(0.1kg)(150 m \cdot s^{-1})}{4.5kg} = -v_{shotgun} = -3.3 m \cdot s^{-1}$$

The shotgun recoils with an initial velocity of about 3 m/s (≈ 7.5 mph). Depending upon how rapidly the stock of the gun comes to rest against the shooter's shoulder, this could be a heck of a wallop. Unless the gun has a padded stock or the shooter is wearing a padded jacked, this would be like getting hit in the shoulder by a 93 mph fastball (an official major league ball has a mass of 5 oz troy or about 0.16 kg).

Collisions/Impulse-Momentum

In physics, a collision is an isolated process in which two or more objects interact in such a manner as to exert a relatively strong force one each other for a relatively short time. There are two primary types of collisions we study in physics, perfectly *elastic* and perfectly *inelastic*. These are two end points along a continuum of possibilities. Because of the mathematical complexity involved in studying collisions that are not perfectly elastic or inelastic we will confine ourselves to the two idealized cases.

Elastic collisions: Elastic collisions occur only between extremely hard objects. In elastic collisions objects exchange momentum by "bouncing" from each other during the collision process (like billiard balls). It is not necessary for any of the interacting particles to actually touch each other in an elastic collision (as when a proton is scattered, elastically from a nucleus). Both momentum and kinetic energy are conserved in elastic collisions. Elastic collisions are of great interest in physics. Accelerating particles to high speeds and colliding them, elastically, with various targets is an oft used method of interrogating atomic scale matter: For a two body system:

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2} \qquad \left(\vec{P}_i = \vec{P}_f\right)$$

$$\frac{1}{2} m_1 {v_{i1}}^2 + \frac{1}{2} m_2 {v_{i2}}^2 = \frac{1}{2} m_1 {v_{f1}}^2 + \frac{1}{2} m_2 {v_{f2}}^2 \qquad \left(KE_i = KE_f\right)$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Inelastic collisions: Inelastic collisions occur when the objects involved in the collision stick together. Momentum is conserved in inelastic collisions but kinetic energy is not. For a two-body system:

$$m_1 v_{i1} + m_2 v_{i2} = \left(m_1 + m_2\right) v_f \qquad \left(\vec{P}_i = \vec{P}_f\right)$$

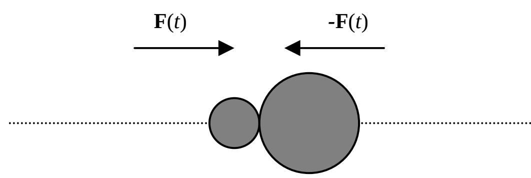
In either case we are generally interested in studying the state of the system before, during, and after the collision process.

Impulse & Momentum

Recall that in an isolated system, the total momentum of the system changes only if some net external force acts on the system:

$$\vec{F} = \frac{\Delta \vec{P}}{\Delta t}$$

Consider:



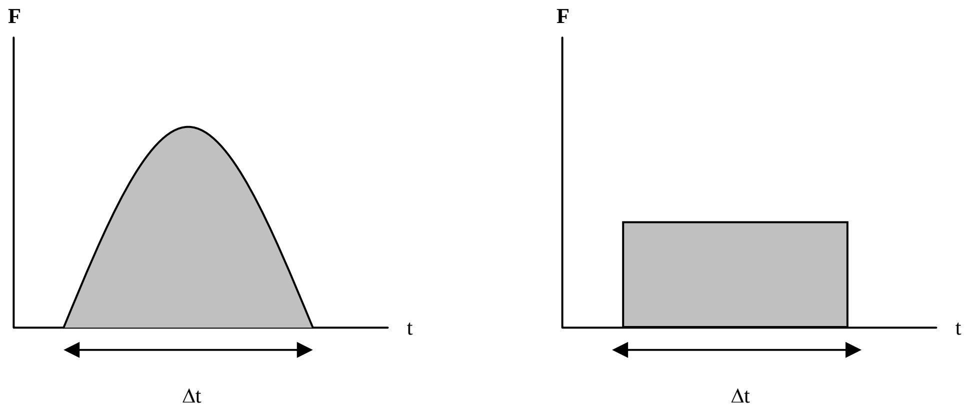
Before the two bodies collide they do not exert force on each other (other than a small gravitational attraction that may be ignored), but as the two bodies collide they do exert force on each other. These forces are not constant, but change in magnitude during the collision process. These forces change the linear momentum of the individual bodies, i.e., the force exerted by the body on the *left* on the body on the *right* changes the linear momentum of the body on the *right*, and vice versa. The amount of this change depends upon both the average value of the forces and the time over which they act. Note that while the individual momenta are changed the overall momentum of the system remains unchanged in this interaction.

If we apply Newton II to either object:

$$\vec{F} = \frac{\Delta \vec{P}}{\Delta t}$$

$$\vec{F}(t)\Delta t = \Delta \vec{P}$$

F(*t*) is a time varying force that with a form similar to the graph below on the left. The graph on the right represents the average value of this same force.



If we average $\vec{F}(t)\Delta t = \Delta\vec{P}$ over the time interval of the collision process (just before, during, and just after), we obtain:

$$\Delta\vec{p} = \vec{F}(t)\Delta t$$

The left side of this equation is Δp, the change in linear momentum of the body being acted upon by the force. The right side contains information about both the magnitude of the force and the duration of the collision and is known, collectively, as the *impulse*, ***J***, of the collision.

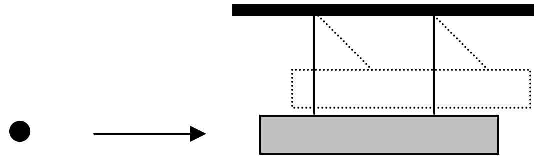
$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{F}(t)\Delta t$$

From this equation it is clear that that the change in linear momentum on a body during a collision process is equal to the impulse that acts on that body. This statement is known as the *impulse-linear momentum theorem*. It is more common to use the average magnitudes of the forces involved in collisions to simplify the mathematics in which case:

$$J = \overline{F}\Delta t$$

Impulse and momentum are both vectors. Unless the bodies colliding conveniently do so in a head-on manner, along an axis, it will be necessary to consider the various components of both quantities.

Example 1. A 0.0025 kg bullet, traveling at a velocity of +425 m/s imbeds itself in the wooden block of a ballistic pendulum. If the wooden block has a mass of 0.200 kg, to what height does the bullet-block combination rise?



Since the bullet and the block "stick together" after the collision, this is an example of a perfectly inelastic collision. Momentum is conserved but kinetic energy is not.

Conserve momentum:

$$mv_{bullet_i} + Mv_{block_i} = (m+M)v_f$$

$$mv_{bullet_i} = (m+M)v_f$$

$$v_f = \frac{mv_{bullet\,i}}{m+M} = 5.25m \cdot s^{-1} = v_{aftercollision}$$

After the collision energy is conserved as long as non-conservative forces do not act on the system, hence:

$$KE_{aftercollision} = PE_f$$

$$\frac{1}{2}(m+M)v_{ac}^{\ 2} = (m+M)gh$$

$$h = \frac{v_{ac}^{\ 2}}{2g} = 1.4m$$

What is the percentage of mechanical energy lost in this problem?

Example 2. A 10 kg ball (m_1) with a velocity of +10 m/s collides head on in an elastic manner with a 5 kg ball (m_2) at rest. What are the velocities after the collision?

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{(10-5)kg}{(10+5)kg}\left(+10m \cdot s^{-1}\right) = +3.33m \cdot s^{-1}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2(10kg)}{(10+5)kg}\left(+10m \cdot s^{-1}\right) = +13.33m \cdot s^{-1}$$

Which ball experiences the greater change in momentum?

Example 3. How good are automobile bumpers? A 1500 kg auto collides elastically with an immovable wall. If the collision lasts 0.150 seconds, the auto was initially traveling +15.0 m/s and rebounds from the wall with a velocity of –2.6 m/s, what is the average force exerted on the bumper?

$$\vec{p}_i = mv_i = \left(2.25 \times 10^4\, kg \cdot m \cdot s^{-1}\right)$$

$$\vec{p}_f = mv_f = -\left(0.39 \times 10^4\, kg \cdot m \cdot s^{-1}\right)$$

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = -\left(2.64 \times 10^4\, kg \cdot m \cdot s^{-1}\right)$$

$$\overline{F} = \frac{\Delta p}{\Delta t} = -\left(1.76 \times 10^5\, N\right)$$

Is momentum conserved in this example?

- What is the significance of the minus sign here?
- Is this an example of conservation of momentum?

Example 4. An 80 kg skier starts from rest at the top of a midwestern ski run. Being new to the sport and unable to control his speed, he quickly finds himself bombing the icy (essentially frictionless) hill. At the base of the hill he is unable to stop. He skis across an icy deck, through the open doors of the ski lodge; across the concrete floor of the rental shop a distance of 10 meters before coming to rest against the back wall of the shop. If the ski hill is 150 meters tall, the deck and lodge are on level ground at the base of the hill, land the unfortunate skier experiences an Impulse of 4250 N·s during the collision, what is the coefficient of kinetic friction between rental skis and concrete?

Impulse-Momentum: $\overline{F}\Delta t = \Delta \vec{p} = m\vec{v}_f - m\vec{v}_i$

$-4250 N \cdot s = -m\vec{v}_i$

$v_i = 53.1 m \cdot s^{-1}$

This is the velocity of the skier just before colliding with the wall. We can use this to compute the energy of the skier just before the collision with the wall:

$$KE = \frac{1}{2}mv^2 = 112784.4J$$

The potential energy of the skier at the top of the hill is:

$$PE = mgh = 117600J$$

The difference in energy is $\Delta E = -4815.6J$. This is the energy that is lost to friction as the skis slide across the concrete floor and is equal to the work done by non-conservative forces, in this case friction. This means that W_{nc} reduces the kinetic energy of the skier. Recall that work equals force x distance. Hence:

$$W_{nc} = \vec{f}_k \cdot \vec{s} = -\mu_k mgs \Rightarrow \frac{4815.6J}{(10m)(80kg)(9.8m \cdot s^{-2})} = \mu_k = 0.61$$

Example 5. Compute the impulse experienced when a 70 kg climber lands on firm ground after falling off a boulder problem from a height of 3 meters.

First let's compute the velocity of the climber before hitting the ground from a height of 3 meters:

$$v_i = \sqrt{2gh} = \sqrt{(2)(9.8m \cdot s^{-2})(3m)} = 7.7m \cdot s^{-1} \therefore \vec{v}_i = -7.7m \cdot s^{-1}$$

The final velocity:

$$v_f = 0$$

Applying Impulse-Momentum:

$$\overline{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = 0 + (70kg)(7.7m \cdot s) = 540N \cdot s$$

Now, if the climber lands stiff-legged and stops in a distance of 1.0 cm, find the average force exerted on them by the ground.

First we compute the average velocity over the course of the collision:

$$\overline{v} = \frac{v_i + v_f}{2} = 3.8m \cdot s^{-1}$$

Next the distance over which the collision occurs:

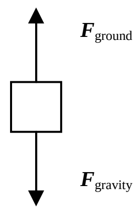
$$d = \overline{v}\Delta t \therefore \Delta t = \frac{d}{\overline{v}} = \frac{0.01m}{3.8m \cdot s^{-1}} = 0.0026s$$

So they come to a complete stop in 0.0026s!

$$\overline{F}\Delta t = \Delta\vec{p} \therefore \overline{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{540N \cdot s}{0.0026s} = 2.1 \times 10^5 N$$

Note that this is a net value (total force). We seek the average force exerted by the ground on the climber. This includes the climber's "weight."

$$\sum \vec{F} = \overline{F}_{ground} - \vec{F}_{fravity}$$



$$\sum \vec{F} = \overline{F}_{ground} - mg$$

$$2.1\times 10^{5}\,N + 690N = \overline{F}_{ground} = 210690N$$

Note that it's not a significant difference.

This force is over 300 times the force exerted by gravity (the person's body weight)! Also note that this is an *average force*. The maximum force exerted is much greater. This is a classic problem in controlling the damage in a collision process, i.e., how to either lengthen the time of the collision or lengthen the distance over which the collision occurs to reduce the impact forces involved. In this particular case this could be accomplished if the person just allowed their legs to flex upon landing. Assume that the climber can flex their legs 0.5 meters. Then:

$$\Delta t = \frac{d}{\overline{v}} = \frac{0.5m}{3.8m\cdot s^{-1}} = 0.13s$$

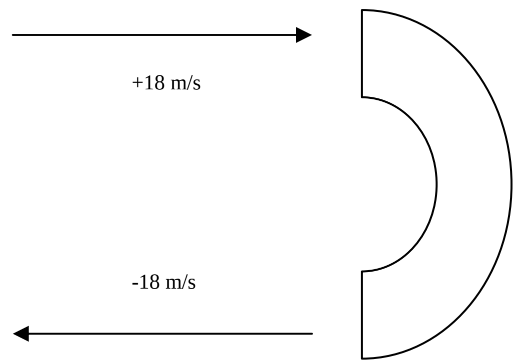
$$\overline{F}\Delta t = \frac{\Delta\vec{p}}{\Delta t} = \frac{540N\cdot s}{0.13s} = 4.2\times 10^{3}\,N$$

$$\overline{F}_{ground} = \sum \vec{F} + mg$$

$$\overline{F}_{ground} = 4.9\times 10^{3}\,N$$

This is about 7x body weight. A vast improvement.

Example 6. A stream of water strikes a stationary turbine blade as shown below, The incident stream of water has a velocity of 18.0 m/s, while the exiting water stream has a velocity of –18.0 m/s. The mass of the water per second that strikes the blade is 25.0 kg/s. Find the magnitude of the average force exerted by the water on the blade.



$$\overline{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = -mv - mv \therefore \overline{F} = \frac{-2mv}{\Delta t} = \frac{(-2)(25.0kg \cdot s^{-1})(18.0m \cdot s^{-1})}{1s} = 900N$$

Example 7. A bullet (m = 0.012kg) is fired into a block (m = 0.100kg) at rest on a rough surface. After the impact the block slides 7.5 meters before coming to rest. If the coefficient of kinetic friction between the block and the surface is 0.65, find the velocity of the bullet before it impacts the block.

The collision between the bullet and the block is inelastic. So we can conserve momentum during the collision process.

$$mv_i + Mv_i = (m + M)v_f$$

$$mv_i = (m + M)v_f$$

Where v_i is the initial velocity of the bullet and v_f is the velocity of the bullet-block system after impact.

The bullet-block combination slides along a rough surface and eventually comes to rest. The entire change in mechanical energy is a change in kinetic energy and is entirely due to the work done by the force of friction. Recall the work = force x distance:

$$\Delta KE = W_{nc} = \vec{f}_k \cdot \vec{d} = -\mu_k mgd \Rightarrow -\frac{1}{2}mv_i^{\ 2} = -\mu_k mgd$$

$$v_i = \sqrt{2(0.65)(9.8m \cdot s^{-2})(7.5meters)} = 9.8m \cdot s^{-1}$$

Here, v_i is the velocity of the bullet-block combination *after* the collision. Noting that v_i here is equal to v_f from the first step:

$$v_i = \frac{m + M}{m}v_f = \frac{.112kg}{.012kg}\left(9.8m \cdot s^{-1}\right) = 91.5m \cdot s^{-1}$$

Example 8. A bullet (*m* = 0.012kg) is fired into a block (*m* = 0.100kg) at rest at the edge of a smooth tabletop. If the tabletop is 1 meter above the floor, and the range of the bullet-block is 2 meters, what is the initial velocity of the bullet?

The collision between the bullet and the block is inelastic. So, as before, we can conserve momentum during the collision process.

$$mv_i + Mv_i = (m + M)v_f$$

$$mv_i = (m + M)v_f$$

$$v_i = \frac{m + M}{m} v_f$$

Since we are interested in the velocity of the bullet-block system after the collision in the *x* direction, it is probably easiest to use kinematics to get what we need. Given a range of 2 meters and a height of 1 meter:

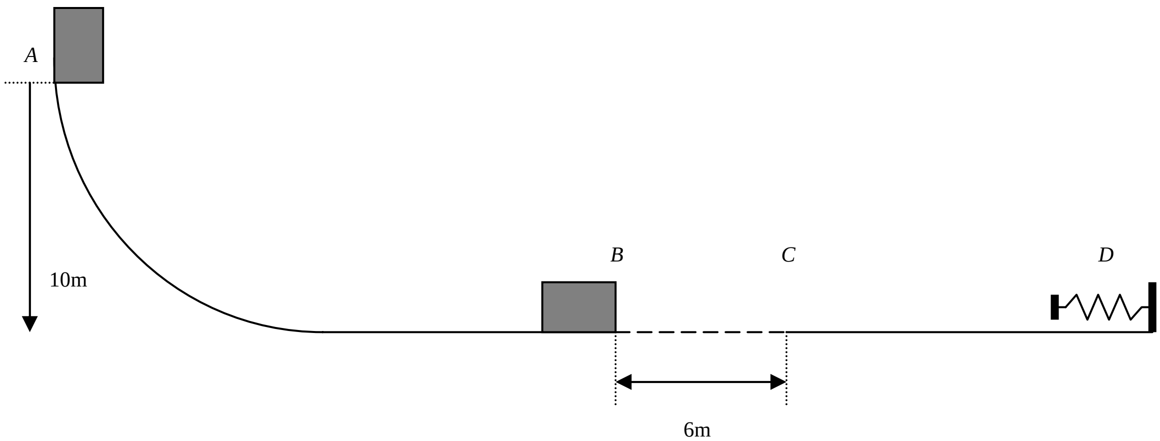
$$t = \sqrt{\frac{2(y - y_0)}{g}} = 0.45s$$

$$\vec{x} = \vec{v}_x t \therefore \frac{\vec{x}}{t} = \vec{v}_x = 4.4m \cdot s^{-1}$$

Where v_i is the velocity of the bullet-block combination *after* the collision. Noting, again, that v_i here is equal to v_f from the first step:

$$v_i = \frac{m + M}{m} v_f = \frac{.112kg}{.012kg}\left(4.4m \cdot s^{-1}\right) = 41.1m \cdot s^{-1}$$

Example 9. A 10 kg block is released from rest at point *A* on track *ABCD* as shown below. The block collides in a perfectly inelastic manner with another block, initially at rest, of mass 5 kg located just before section *BC*. The track is smooth except for section *BC*, of length 6 meters. The blocks hit the spring (k = 4000 N/m) and compress it a distance of 0.3 meters from its equilibrium position. Determine the coefficient of kinetic friction between the blocks and section *BC* of the track.



We have a combination of four events here: one in which mechanical energy is conserved, an inelastic collision, one in which work done by a non-conservative force changes the kinetic energy of the system, and a conversion of kinetic energy to elastic potential energy. Because of the third event (work done by a non-conservative force) the final (elastic) energy of the system is less than the initial (potential) energy of the system by an amount equal to the work done by the non-conservative force of friction.

Consider what is conserved (or not) at each step along the way: during the initial motion of the system, during the collision, during the movement across the rough surface, from the end of the rough surface to the spring, during compression of the spring.

Event 1

From conservation of mechanical energy, the velocity of the first block just before it's collision with the second is:

$$v = \sqrt{2gh} = 14m \cdot s^{-1}$$

Event 2

Before and after the inelastic collision between the blocks:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

$$m_1 v_i = (m_1 + m_2) v_f$$

$$v_f = \frac{m_1}{m_1 + m_2} v_i \Rightarrow \frac{10kg}{15kg} 14 m \cdot s^{-1} = 9.3 m \cdot s^{-1}$$

Events 3 & 4

Since we know the final energy of the system and the velocity of both blocks as they begin to move across the rough section we can compute the energy lost due to the force of friction, note that this is equal to the work done by the force of friction, then noting that work = force x distance:

$$W_{nc} = E_f - E_i = \vec{f}_k \cdot d$$

Hence:

$$\vec{f}_k \cdot \vec{d} = \frac{1}{2} kx^2 - \frac{1}{2}(m_1 + m_2) v_f^{\,2} = -\mu_k (m_1 + m_2) gd = \therefore \mu_k = \frac{\frac{1}{2}\left(kx^2 - (m_1 + m_2) v_f^{\,2}\right)}{(m_1 + m_2) gd} = 0.531$$

Example 10. Compute the average force exerted by foot pegs on a 70 kg motorcyclist when they "stick" a landing on flat ground from a height of 3 meters if they are able to flex their legs 50 cm in the process. Compare this to a smooth landing that slopes downward at an angle of 45^0 (i.e., the bike continues at an angle of 45^0 downward after touchdown). In both cases assume that the component of velocity in the *x* direction just before impact is negligible.

If the landing is flat then the velocity after impacting the ground is zero.

$$v_i = \sqrt{2gh} = \sqrt{(2)(9.8m \cdot s^{-2})(3m)} = 7.7m \cdot s^{-1} \therefore \vec{v}_i = -7.7m \cdot s^{-1}$$

$$v_f = 0$$

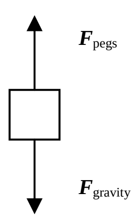
$$\overline{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = 0 + (70kg)(7.7m \cdot s) = 540N \cdot s$$

$$\overline{v} = \frac{v_i + v_f}{2} = 3.8m \cdot s^{-1}$$

$$d = \overline{v}\Delta t \therefore \Delta t = \frac{d}{\overline{v}} = \frac{0.50m}{3.8m \cdot s^{-1}} = 0.13s$$

$$\overline{F}\Delta t = \frac{\Delta\vec{p}}{\Delta t} = \frac{540N \cdot s}{0.13s} = 4.2 \times 10^3 N$$

$$\overline{F}_{pegs} = \sum \vec{F} + mg$$



$$\overline{F}_{pegs} = 4.9 \times 10^3 N$$

If the landing slopes downward and is smooth then the velocity after impacting the ground is not zero. The force that the ground exerts on the motorcycle causes a change in direction.

$$v_i = \sqrt{2gh} = \sqrt{(2)(9.8m \cdot s^{-2})(3m)} = 7.7m \cdot s^{-1} \therefore \vec{v}_i = -7.7m \cdot s^{-1}$$

$$v_{ac} = v_f \sin 45^0 = -5.4m \cdot s^{-1}$$

$$\bar{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = (70kg)(-5.4m\cdot s^{-1}) + (70kg)(7.7m\cdot s^{-1}) = 161N\cdot s$$

$$\bar{v} = \frac{v_i + v_f}{2} = 6.6m\cdot s^{-1}$$

$$d = \bar{v}\Delta t \therefore \Delta t = \frac{d}{\bar{v}} = \frac{0.50m}{6.6m\cdot s^{-1}} = 0.076s$$

$$\bar{F}\Delta t = \frac{\Delta\vec{p}}{\Delta t} = \frac{161N\cdot s}{0.076s} = 2.1\times 10^3 N$$

$$\bar{F}_{pegs} = \sum\vec{F} + mg$$

$$\bar{F}_{pegs} = 2.8\times 10^3 N$$

Or about half the impact force suffered in a flat landing!

Although we've been able to compute landing vectors before in problems like this, we've never been able to consider what happens *after* the moment just before impact. This technique is a powerful new tool in our arsenal.